Climbing the Potential Vorticity Staircase: How Profile Modulations Nucleate Profile Structure and Transport Barriers

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Thanks for:

- Collaborations: M. Malkov, A. Ashourvan, Y. Kosuga, O.D. Gurcan, D.W. Hughes
- Discussions: G. Dif-Pradalier, Z.B. Guo, P.-C. Hsu, W.R. Young, J.-M. Kwon

Outline

A) A Primer on "Tokamak Plasma" Turbulence, Zonal Flows and Modulational Instability

I) Systems:

- "Tokamak Plasma" Primer

II) Mesoscopic Patterns

- Avalanches
- Zonal Flows – via modulation of the gas of drift waves

B) Pattern competition – Enter the staircase!

C) The Basics: QG staircase

- Model content
- Results and FAQ's

Outline, cont'd

D) The H.-W. staircase: profile structure and barrier formation

- extending the model
- profile formation
- transport bifurcation

F) Lessons, Conclusions, Future

I) The System:
What is a Tokamak?

How does confinement work?

N.B. No programmatic advertising intended…

Magnetically confined plasma

- Nuclear fusion: option for generating large amounts of carbon-free energy
- Challenge: ignition -- reaction release more energy than the input energy

  Lawson criterion:

$$n_i\tau_E T_i > 3\times10^{21}\mathrm{m}^{-3}\mathrm{s\,keV}$$

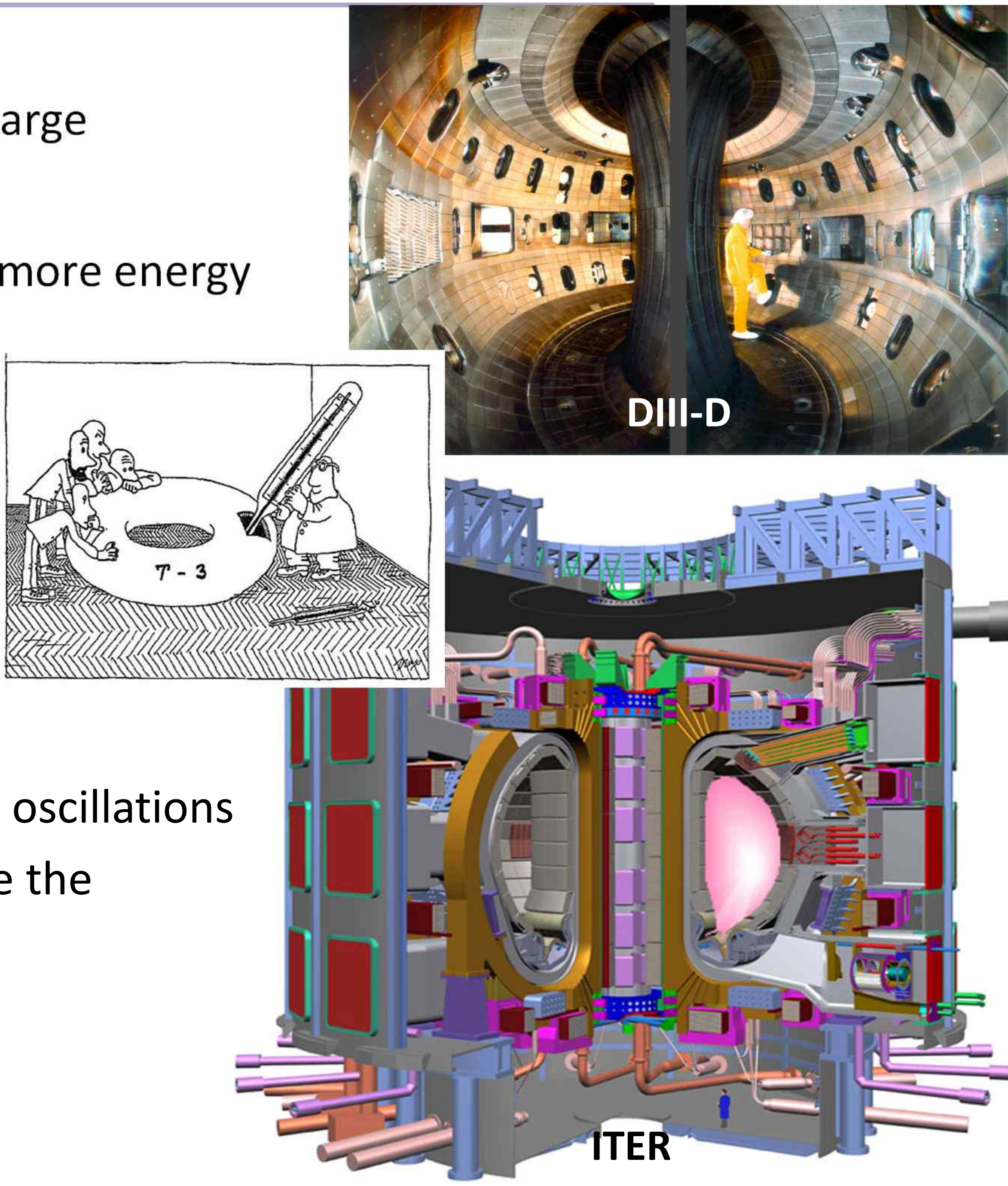
  → confinement

  → turbulent transport

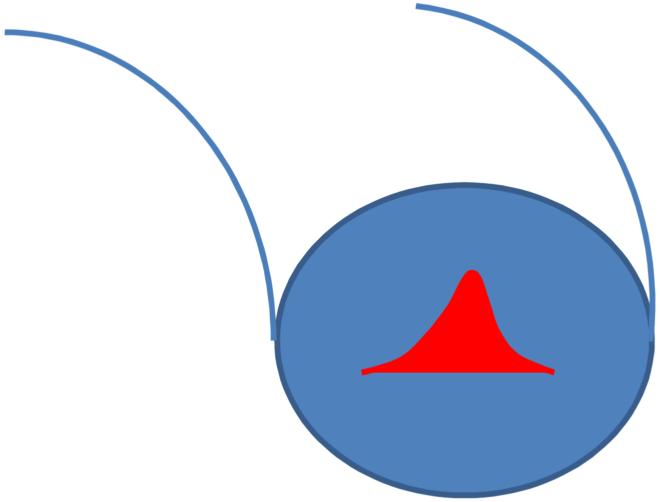
- Turbulence: instabilities and collective oscillations

  → lowest frequency modes dominate the transport

  → drift wave

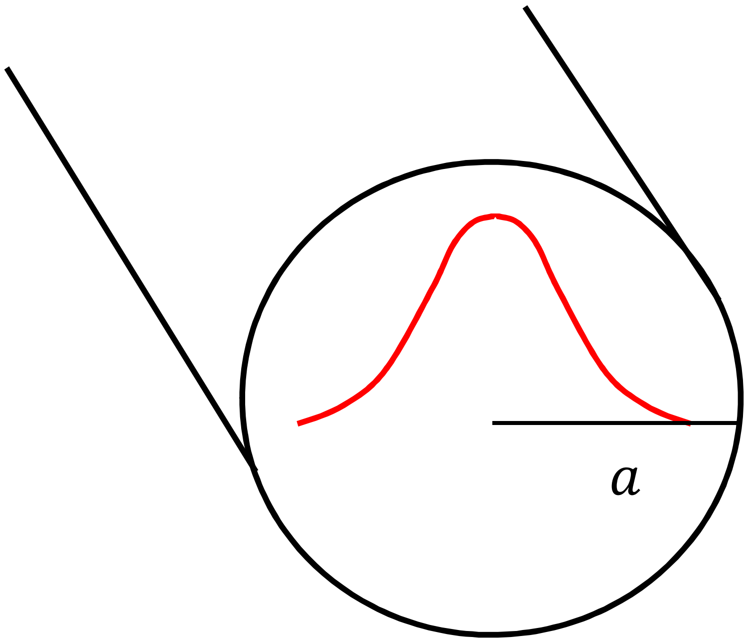


Primer on Turbulence in Tokamaks I



- Strongly magnetized
  - Quasi 2D cells
  - Localized by $\vec{k} \cdot \vec{B} = 0$ (resonance)
- $\vec{V}_{\perp} = +\frac{c}{B}\,\vec{E} \times \hat{z}$
- $\nabla T_e, \nabla T_i, \nabla n$ driven
- Akin to thermal Rossby wave, with: g → magnetic curvature
- Resembles wave turbulence, not high Re Navier-Stokes turbulence
- Re ill defined, "Re" ≤ 100
- , $K \sim \tilde{V}\tau_c/\Delta \sim 1$ → *Kubo #* ≈ 1
- Broad dynamic range → multi-scale problem: $a, L_P, \Delta r_c, \rho_i, \Delta r_{c_e}, \rho_e$

Primer on Turbulence in Tokamaks II



- Characteristic scale ~ few ρ_i → "mixing length"
- Characteristic velocity $v_d \sim \rho_* c_s$

2 scales:
$\rho \equiv$ gyro-radius
$a \equiv$ cross-section
$\rho_* \equiv \rho / a$ ➔ key ratio

Gyro-Bohm Bohm

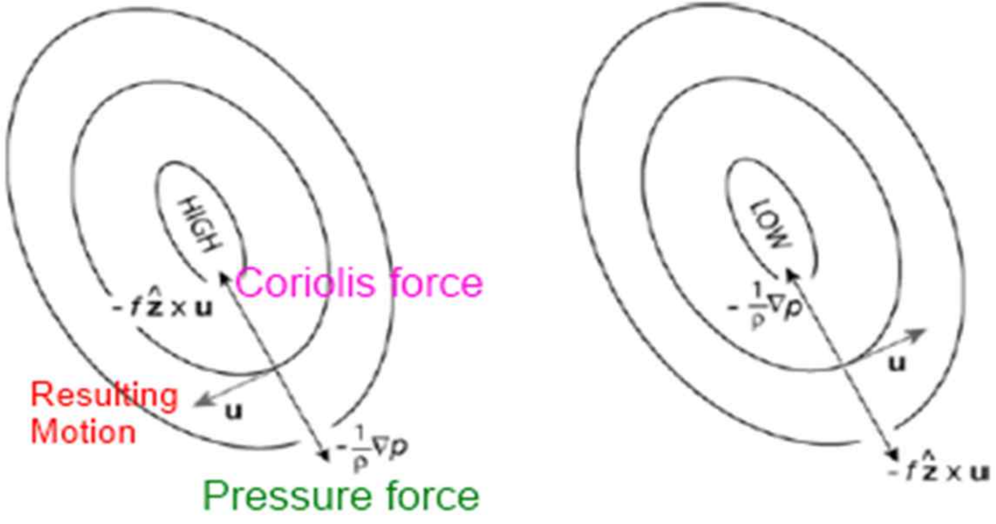
- Transport scaling: $D \sim \rho v_d \sim \rho_* D_B \sim D_{GB}, \quad D_B \sim \rho_s c_s$
- i.e. Bigger is better! ➔ sets profile scale via heat balance (Why ITER is enormous…)
- Reality: $D \sim \rho_*^{\alpha} D_B$, $\alpha < 1$ ➔ why?? – pattern competition?
- 2 Scales, $\rho_* \ll 1$ ➔ key contrast to familiar pipe flow

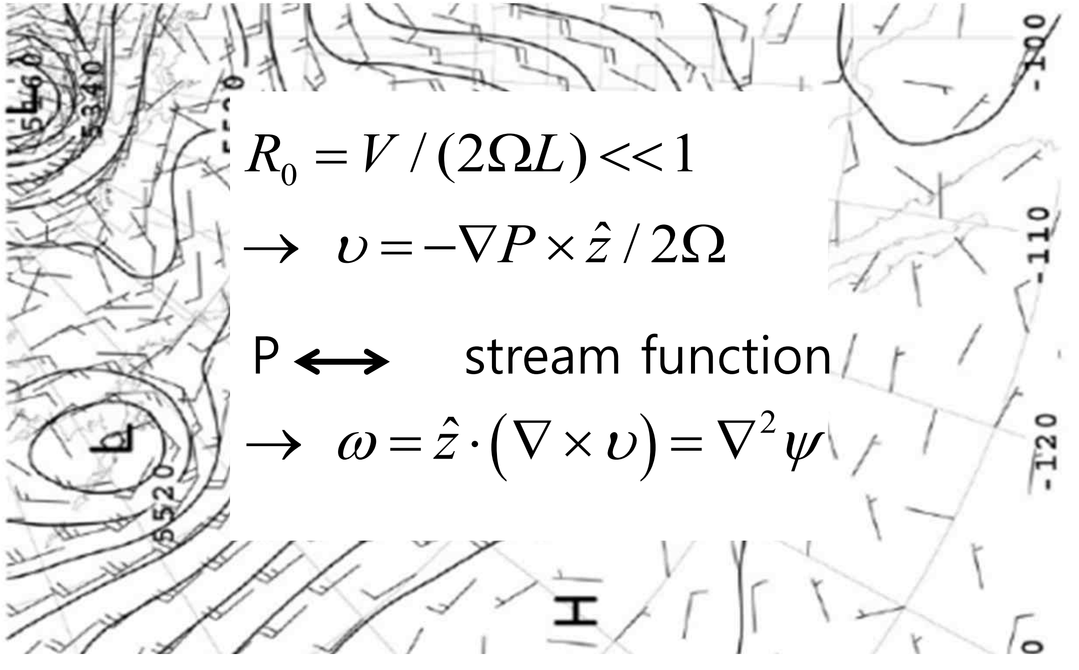
Model: GFD-Plasma Duality (Hasegawa, et. seq.)

Geophysical fluids

- Phenomena: weather, waves, large scale atmospheric and oceanic circulations , water circulation, jets…
- Geophysical fluid dynamics (GFD): low frequency ($\omega < \Omega$)

  “We might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician more of the Beethoven than the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand.” – J.G. Charney (“Turing’s Cathedral”)
- Geostrophic motion: balance between the Coriolis force and pressure gradient





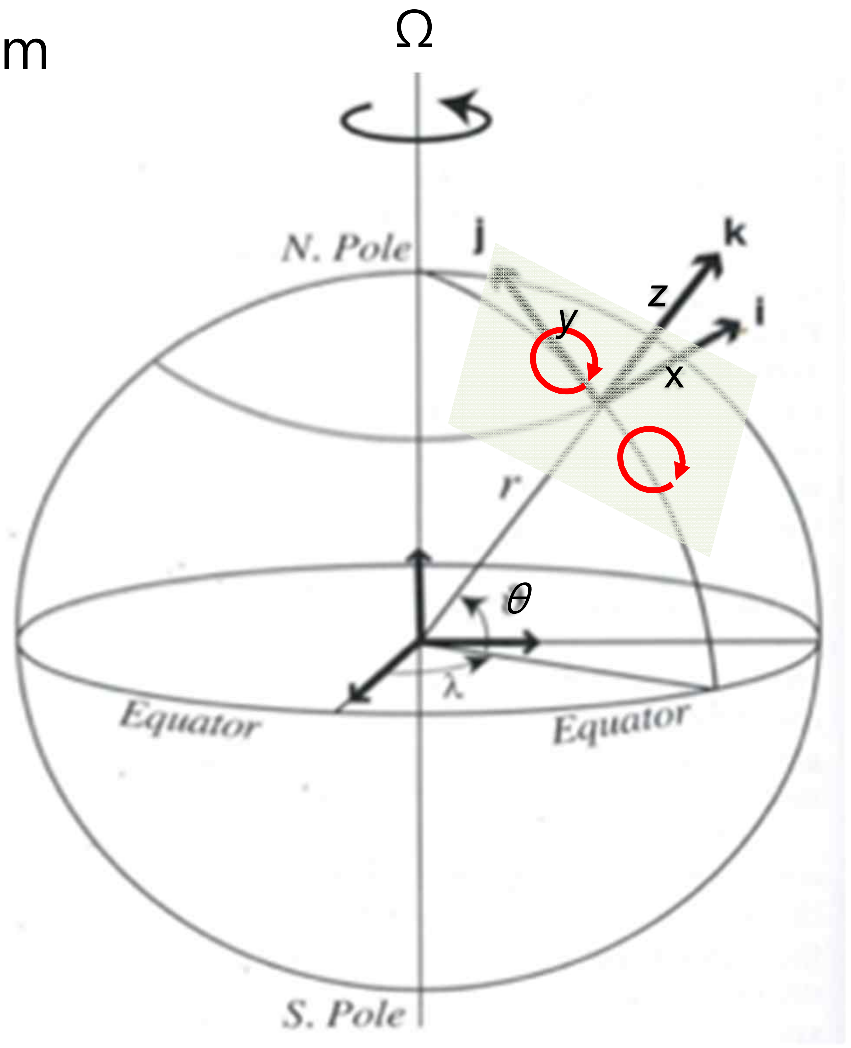
$R_0 = V/(2\Omega L) << 1$

$\rightarrow \ \upsilon = -\nabla P \times \hat{z}/2\Omega$

P ⟷ stream function

$\rightarrow \ \omega = \hat{z}\cdot\left(\nabla\times\upsilon\right) = \nabla^2\psi$

Kelvin's theorem – unifying principle



- Kelvin's circulation theorem for rotating system

$$\oint v \cdot dl = \int (\underbrace{\nabla \times v}_{\text{relative}} + \underbrace{2\Omega}_{\text{planetary}}) \cdot \hat{z}\, dS \equiv C \qquad \dot{C} = 0$$

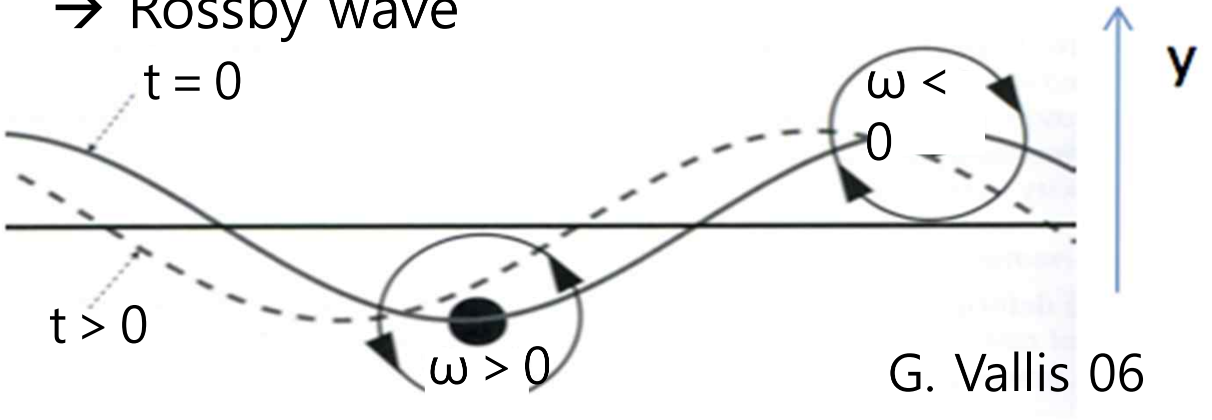
- Displacement on beta-plane

$$\dot{C} = 0 \;\rightarrow\; \frac{d}{dt}\nabla^2\psi = -2\Omega\cos\theta\frac{d\theta}{dt} = -\underbrace{\beta}_{\beta = 2\Omega\cos\theta_0 / R_\oplus} v_y$$

- Quasi-geostrophic eq

$$\boxed{\frac{d}{dt}\left(\nabla^2\psi + \beta y\right) = 0 \;\; \textbf{PV conservation}}$$

→ Rossby wave



G. Vallis 06

Drift wave model – Fundamental prototype

- Hasegawa-Wakatani : simplest model incorporating instability

$$V = \frac{c}{B}\hat{z}\times\nabla\phi + V_{pol}$$

$$J_\perp = n|e|V^i_{pol} \qquad \eta J_\parallel = -\nabla_\parallel\phi + \nabla_\parallel p_e$$

$$\nabla_\perp\cdot J_\perp + \nabla_\parallel J_\parallel = 0$$

→ vorticity: $\rho_s^2 \frac{d}{dt}\nabla^2\phi = -D_\parallel\nabla_\parallel^2(\phi-n)+\nu\nabla^2\nabla^2\phi$

$$\frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0$$

→ density: $\frac{d}{dt}n = -D_\parallel\nabla_\parallel^2(\phi-n)+D_0\nabla^2 n$

→ **PV conservation in inviscid theory** $\frac{d}{dt}\left(n-\nabla^2\phi\right)=0$

→ PV flux = particle flux + vorticity flux

→ zonal flow being a counterpart of particle flux

QL: $\frac{\partial}{\partial t}\langle n\rangle = -\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\tilde{n}\rangle$

→? $\frac{\partial}{\partial t}\langle\nabla^2\phi\rangle = -\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\nabla^2\tilde{\phi}\rangle$

$= -\frac{\partial^2}{\partial r^2}\langle\tilde{\upsilon}_r\tilde{\upsilon}_\theta\rangle$

- Hasegawa-Mima ($D_\parallel k_\parallel^2/\omega >> 1 \;\rightarrow\; n\sim\phi$)

$$\frac{d}{dt}\left(\phi-\rho_s^2\nabla^2\phi\right)+\upsilon_*\partial_y\phi=0$$

PV conservation $\frac{dq}{dt}=0$

GFD: Quasi-geostrophic system	Plasma: Hasegawa-Wakatani system
$q=\nabla^2\psi+\beta y$ relative vorticity planetary vorticity	$q=n-\nabla^2\phi$ density (guiding center) ion vorticity (polarization)
Physics: $\Delta y\rightarrow\Delta\left(\nabla^2\psi\right)$ →ZF	Physics: $\Delta r\rightarrow\Delta n\rightarrow\Delta\left(\nabla^2\phi\right)$ →ZF!

- Charney-Haswgawa-Mima equation

$$n=n_0+\tilde{n}$$

$$\tilde{n}\sim\frac{e\tilde{\phi}}{T}$$

H-W → H-M:

$$\frac{1}{\omega_{ci}}\frac{\partial}{\partial t}\left(\nabla^2\phi-\rho_s^{-2}\phi\right)-\frac{1}{L_n}\frac{\partial}{\partial y}\phi+\frac{\rho_s}{L_n}J(\phi,\nabla^2\phi)=0$$

Q-G:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi-L_d^{-2}\psi\right)+\beta\frac{\partial}{\partial x}\psi\ +\ J(\psi,\nabla^2\psi)=0$$

II) Mesoscopic Patterns in Tokamak Turbulence

→ Avalanches and ‘Non-locality’
→ Zonal Flows

→ "Truth is never pure and rarely simple" (Oscar Wilde)

Transport: Local or Non-local?

- 40 years of fusion plasma modeling
  - local, diffusive transport

$$Q = -n\chi(r)\nabla T, \quad \chi \leftrightarrow D_{GB}$$

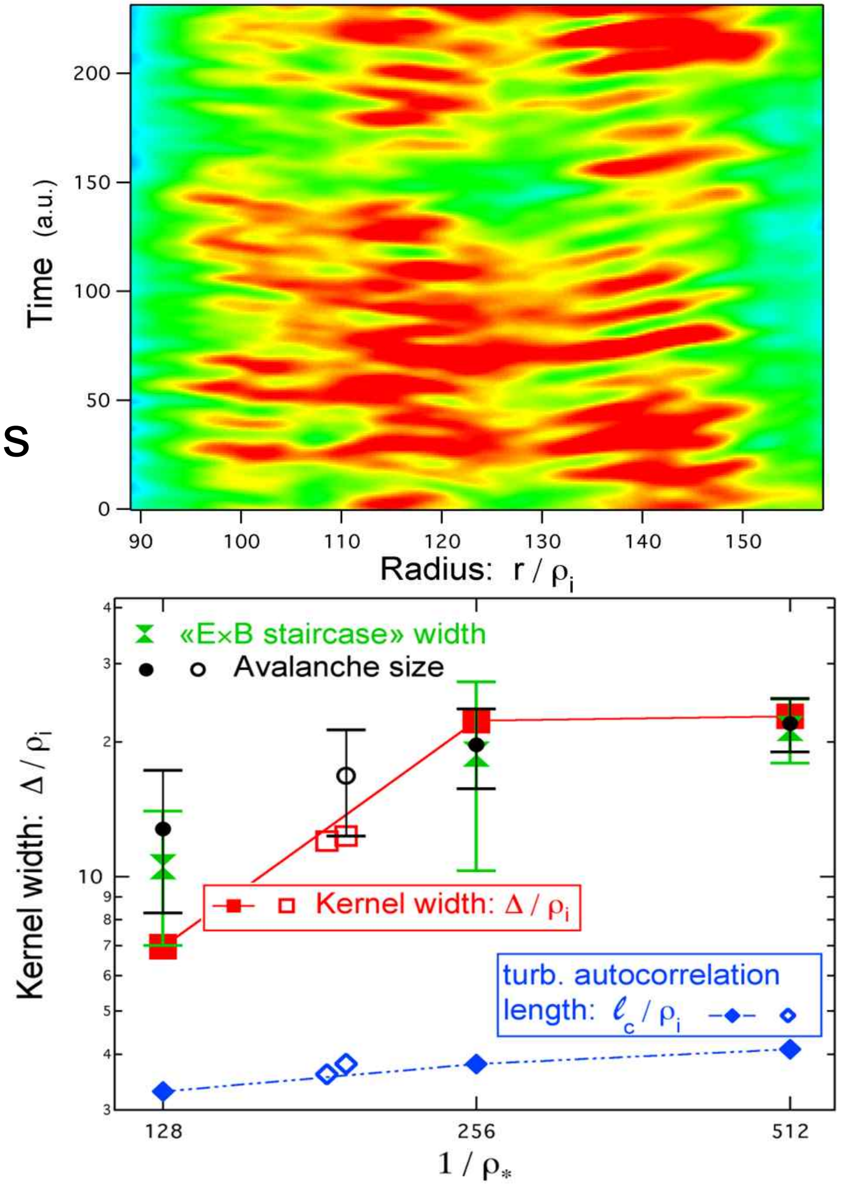
- 1995 → increasing evidence for:
  - transport by avalanches, as in sand pile/SOCs
  - turbulence propagation and invasion fronts
  - "non-locality of transport"

$$Q = -\int \kappa(r,r')\nabla T(r')dr'$$

$$\kappa(r,r') \sim S_0 / \left[(r-r')^2 + \Delta^2\right]$$

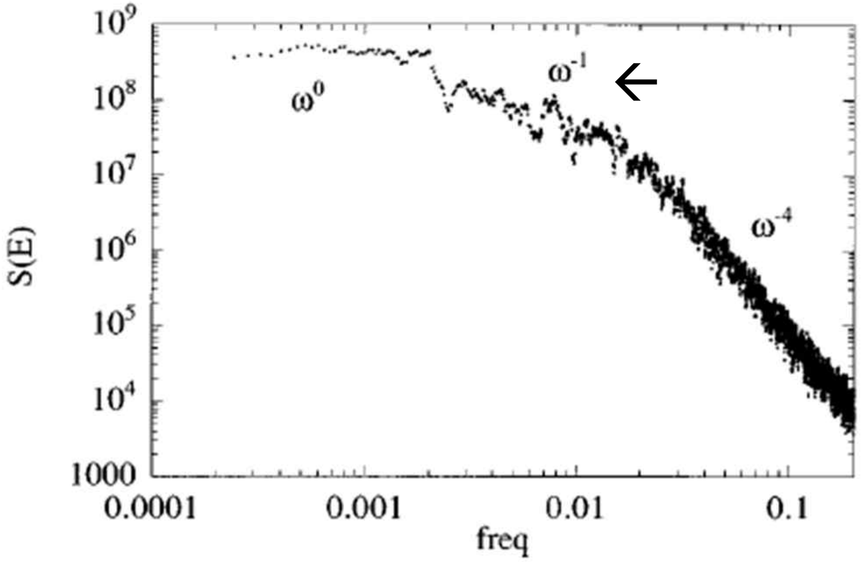
- Physics:
  - Levy flights, SOC, turbulence fronts…
- Fusion:
  - gyro-Bohm breaking
    (ITER: significant ρ_* extension)

→ fundamentals of turbulent transport modeling??

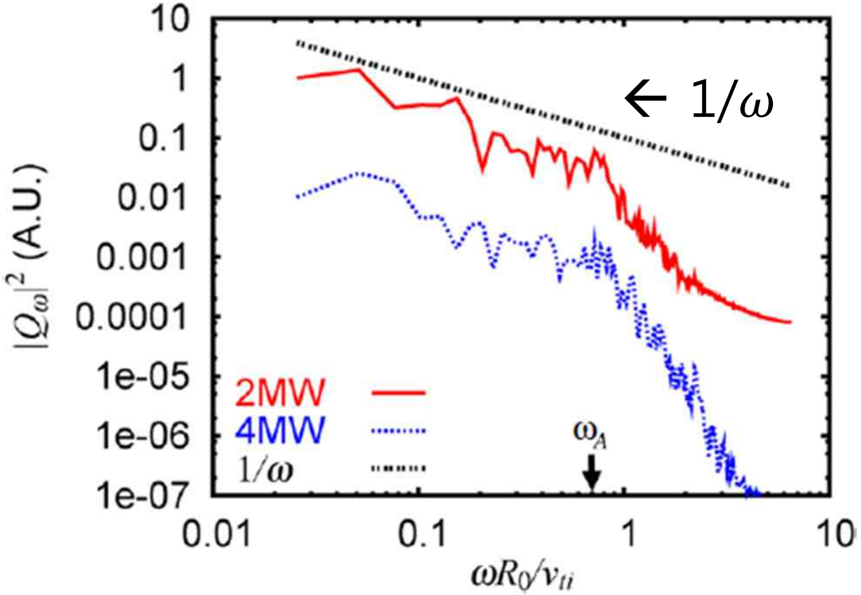


Guilhem Dif-Pradalier et al. PRL 2009

- ‘Avalanches’ form! – flux drive + geometrical ‘pinning’



Newman PoP96 (sandpile)
(Autopower frequency spectrum of ‘flip’)



GK simulation also exhibits avalanching
(Heat Flux Spectrum) (Idomura NF09)

- Avalanching is a likely cause of ‘gyro-Bohm breaking’ → Intermittent Bursts

→ localized cells self-organize to form transient, extended transport events

- Akin domino toppling:
- Pattern competition

  with shear flows!

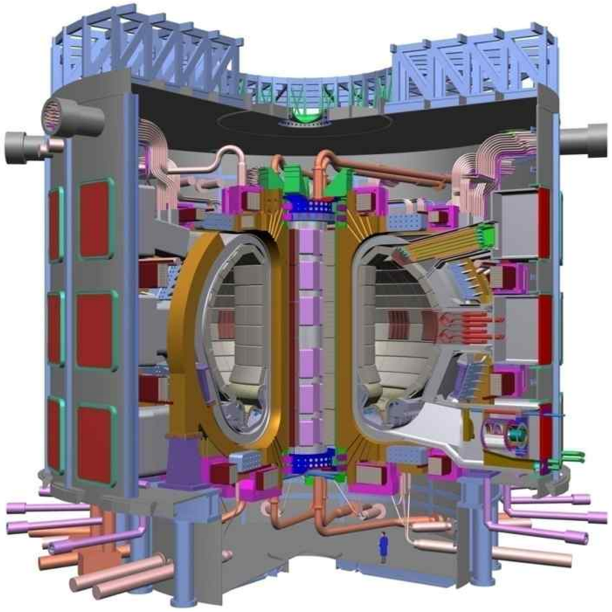
Toppling front can
penetrate beyond region
of local stability

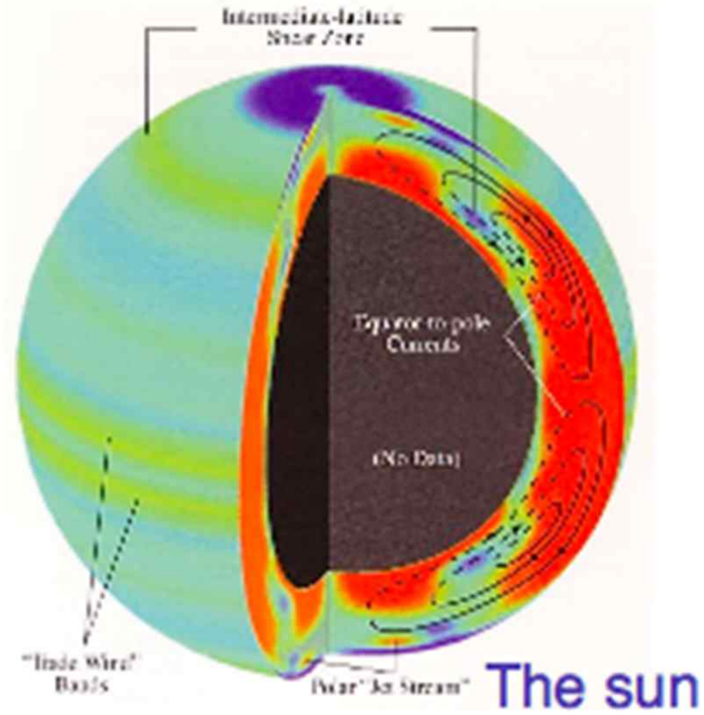
What regulates radial extent? ➔ Shear Flows ‘Natural’ to Tokamaks

- Zonal Flows Ubiquitous for:
  - ~ 2D fluids / plasmas $R_0 < 1$

    Rotation $\vec{\Omega}$, Magnetization $\vec{B}_0$, Stratification

  Ex: MFE devices, giant planets, stars…

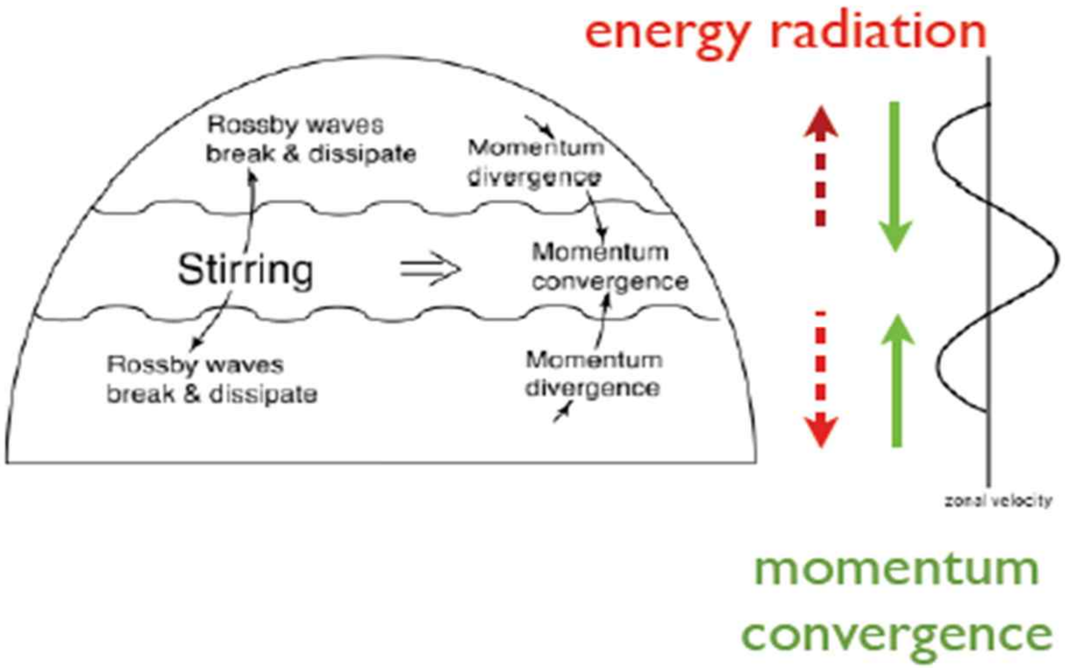




Heuristics of Zonal Flows a): How Form?

Simple Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow (c.f. Vallis '07, Held '01)
- Key Physics:



Rossby Wave:

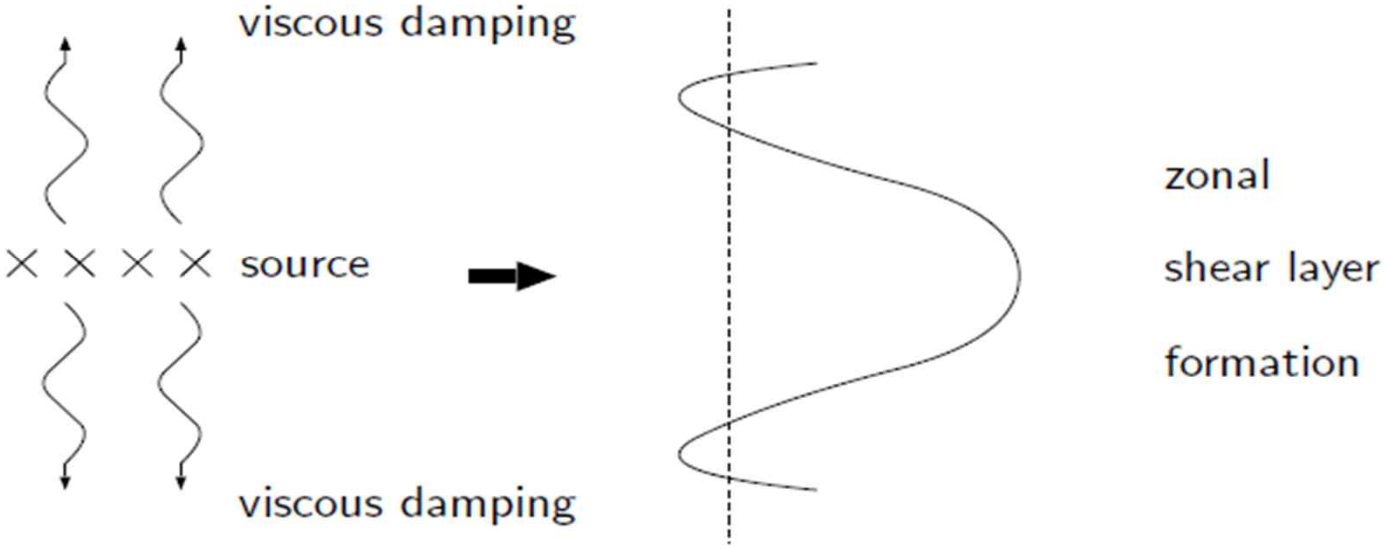
$$\omega_k = -\frac{\beta k_x}{k_\perp^2}$$

$$v_{gy} = 2\beta \frac{k_x k_y}{\left(k_\perp^2\right)^2}\ ,\ \langle \tilde{v}_y \tilde{v}_x \rangle = \sum_{\vec{k}} -k_x k_y \left|\hat{\phi}_{\vec{k}}\right|^2$$

$\therefore\ v_{gy} v_{phy} < 0$ → Backward wave!

➔ Momentum convergence at stirring location

- ..."the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region." (I. Held, '01)
- Outgoing waves ⇒ incoming wave momentum flux

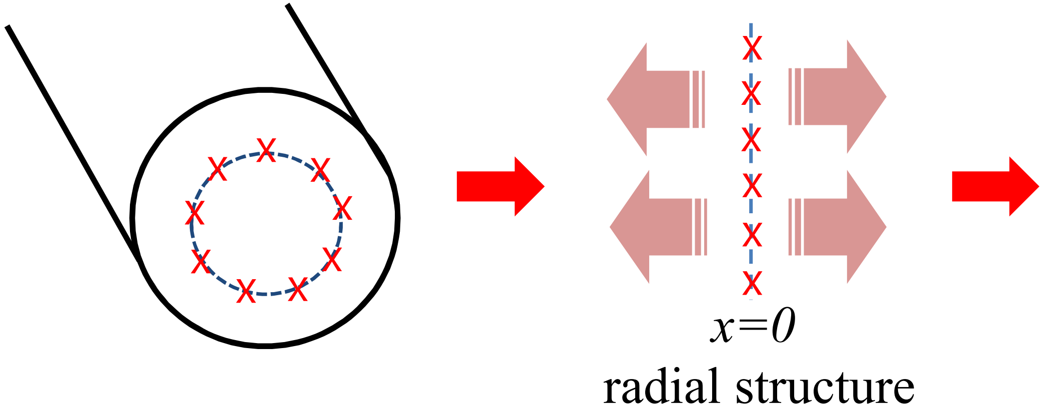


- Local Flow Direction (northern hemisphere):
  - eastward in source region
  - westward in sink region
  - set by $\beta > 0$
  - Some similarity to spinodal decomposition phenomena
    → Both 'negative diffusion' phenomena

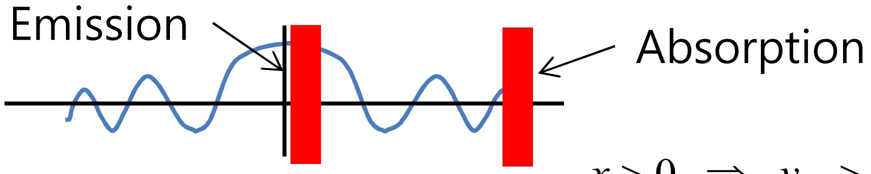
Wave-Flows in Plasmas

MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure



- couple to damping ↔ outgoing wave



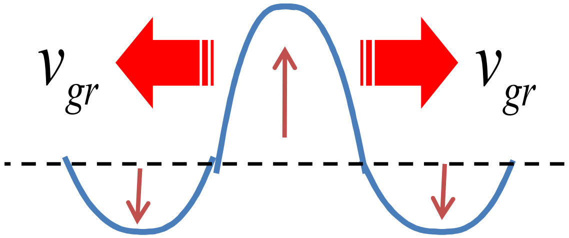
$x > 0 \Rightarrow v_{gr} > 0$

- $v_{gr} = -2\rho_s^2 \frac{k_\theta k_r v_*}{(1+k_\perp^2 \rho_s^2)^2}$

$v_* < 0\,,\ \ k_r k_\theta > 0$

$$\langle v_{rE} v_{\theta E} \rangle = -\frac{c^2}{B^2} |\phi_{\vec{k}}|^2 \, k_r k_\theta < 0$$

- outgoing wave energy flux → incoming wave momentum flux → counter flow spin-up!



- zonal flow layers form at excitation regions

Zonal Flows I

- What is a Zonal Flow?
  - $n = 0$ potential mode; $m = 0$ (ZFZF), with possible sideband (GAM)
  - toroidally, poloidally symmetric *ExB* shear flow
- Why are Z.F.'s important?
  - Zonal flows are secondary (nonlinearly driven):
    - modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. '78)
    - modes of minimal damping (Rosenbluth, Hinton '98)
    - drive zero transport ($n = 0$)
  - natural predators to feed off and retain energy released by gradient-driven microturbulence

Zonal Flows II

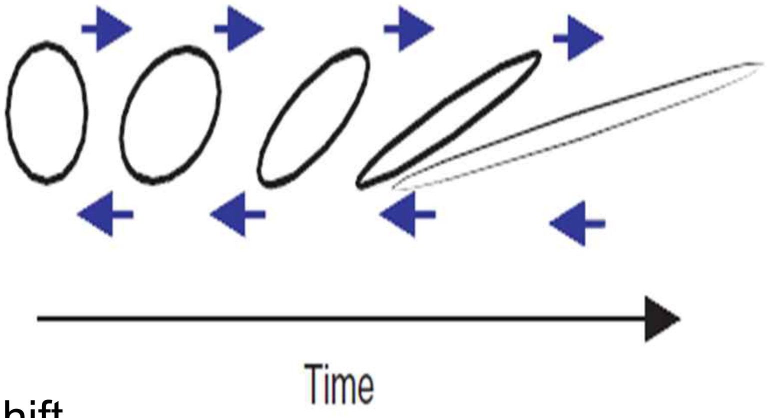
- Fundamental Idea:
  - Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  - Kelvin's theorem is ultimate foundation

- G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  - Polarization charge ➡ $-\boxed{\rho^2}\nabla^2\phi = \boxed{n_{i,GC}(\phi) - n_e(\phi)}$
    polarization length scale ↲ ↳ *ion GC* ↳ *electron density*
  - so $\Gamma_{i,GC} \neq \Gamma_e$ ➡ $\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle \neq 0$ ⬌ 'PV transport'
    ↳ *polarization flux* → What sets cross-phase?
  - If 1 direction of symmetry (or near symmetry):

$$-\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle = -\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$$ (Taylor, 1915)

$-\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ ➡ Reynolds force ➡ Flow

Zonal Flows Shear Eddys I



- Coherent shearing: (Kelvin, G.I. Taylor, Dupree'66, BDT'90)
  - radial scattering + $\langle V_E \rangle'$ → hybrid decorrelation
  - $k_r^2 D_\perp \rightarrow \quad (k_\theta^2 \langle V_E \rangle'^2 D_\perp / 3)^{1/3} = 1/\tau_c$
  - Akin shear dispersion
  - shaping, flux compression: Hahm, Burrell '94

- Other shearing effects (linear):
  - spatial resonance dispersion:

Response shift
and dispersion

$$\omega - k_\parallel v_\parallel \Rightarrow \omega - k_\parallel v_\parallel - k_\theta \langle V_E \rangle' (r - r_0)$$

  - differential response rotation → especially for kinetic curvature effects

→ N.B. Caveat: Modes can adjust to weaken effect of external shear

(Carreras, et. al. '92; Scott '92)

Shearing II

- Zonal Shears: Wave kinetics (Zakharov et. al.; P.D. et. al. '98, et. seq.)

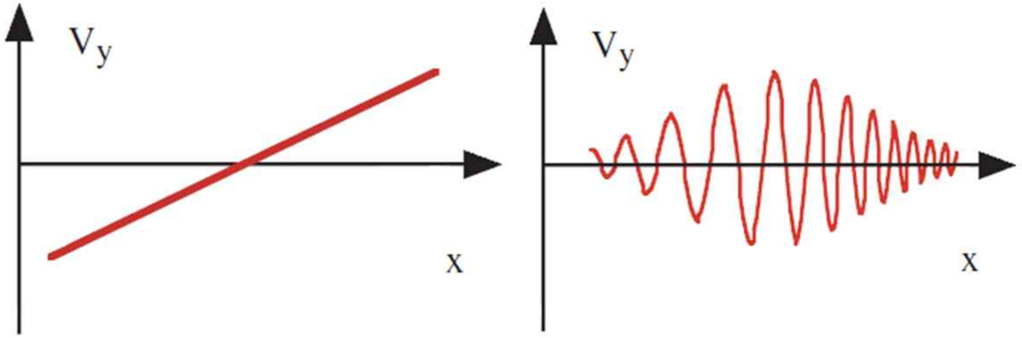
  Coherent interaction approach (L. Chen et. al.)

- $dk_r / dt = -\partial(\omega + k_\theta V_E) / \partial r \, ; \; V_E = \langle V_E \rangle + \tilde{V}_E$

  Mean shearing $: k_r = k_r^{(0)} - k_\theta V_E' \tau$

  Zonal Random shearing $: \langle \delta k_r^2 \rangle = D_k \tau$

  $$D_k = \sum_q k_\theta^2 \left| \tilde{V}'_{E,q} \right|^2 \tau_{k,q}$$



- Wave ray chaos (not shear RPA) underlies $D_k \rightarrow$ induced diffusion
- Induces wave packet dispersion
- Applicable to ZFs and GAMs

- Mean Field Wave Kinetics

$$\frac{\partial N}{\partial t} + (\vec{V}_{gr} + \vec{V}) \cdot \nabla N - \frac{\partial}{\partial r}(\omega + k_\theta V_E) \cdot \frac{\partial N}{\partial \vec{k}} = \gamma_{\vec{k}} N - C\{N\}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N \rangle = \gamma_{\vec{k}} \langle N \rangle - \langle C\{N\} \rangle}$$

Zonal shearing → computed using modulational response

Shearing III

- Energetics: Books Balance for Reynolds Stress-Driven Flows!
- Fluctuation Energy Evolution – Z.F. shearing

$$\int d\vec{k}\,\omega\left(\frac{\partial}{\partial t}\langle N\rangle-\frac{\partial}{\partial k_r}D_k\frac{\partial}{\partial k_r}\langle N\rangle\right)\Rightarrow\frac{\partial}{\partial t}\langle\varepsilon\rangle=-\int d\vec{k}\,V_{gr}(\vec{k})D_{\vec{k}}\frac{\partial}{\partial k_r}\langle N\rangle \qquad V_{gr}=\frac{-2k_r k_\theta V_* \rho_s^2}{\left(1+k_\perp^2\rho_s^2\right)^2}$$

Point: For $d\langle\Omega\rangle/dk_r<0$, Z.F. shearing damps wave energy

- Fate of the Energy: Reynolds work on Zonal Flow

Modulational Instability

$$\partial_t\delta V_\theta+\partial\left(\delta\langle\tilde{V}_r\tilde{V}_\theta\rangle\right)/\partial r=-\gamma\delta V_\theta$$

$$\delta\langle\tilde{V}_r\tilde{V}_\theta\rangle\sim\frac{k_r k_\theta\delta\Omega}{(1+k_\perp^2\rho_s^2)^2}$$

N.B.: Wave decorrelation essential:
Equivalent to PV transport
(c.f. Gurcan et. al. 2010)

Modulation → inhomogeneity
in PV mixing

- Bottom Line:
  - Z.F. growth due to shearing of waves
  - “Reynolds work” and “flow shearing” as relabeling → books balance
  - Z.F. damping emerges as critical; MNR ‘97

Approaches to Modulation

~ Weak, Wave Turbulence Problems

→ Quasi-particle, Wave Kinetics → δN

See: P.D. Itoh, Itoh, Hahm '05 PPCF

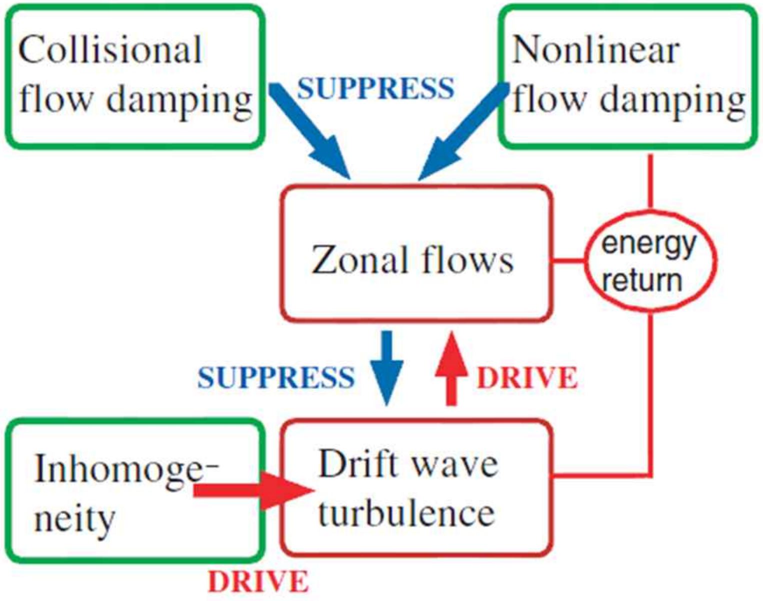
→ Envelope Theory, Generalized NLS → ψ

See: O.D. Gurcan, P.D. '2014 J. Phys. A.

N.B.: Representation of PV mixing and its inhomogeneity is crucial

Feedback Loops I

- Closing the loop of shearing and Reynolds work



- Spectral ‘Predator-Prey’ equations

Prey → Drift waves, $\langle N \rangle$

$$\frac{\partial}{\partial t}\langle N\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle = \gamma_k \langle N\rangle - \frac{\Delta\omega_k}{N_0}\langle N\rangle^2$$

Predator → Zonal flow, $|\phi_q|^2$

$$\frac{\partial}{\partial t}|\phi_q|^2 = \Gamma_q\left[\frac{\partial\langle N\rangle}{\partial k_r}\right]|\phi_q|^2 - \gamma_d|\phi_q|^2 - \gamma_{NL}[|\phi_q|^2]|\phi_q|^2$$

Feedback Loops II

- Recovering the 'dual cascade':
  - Prey → <N> ~ <Ω> ⇒ induced diffusion to high k_r { ⇒ Analogous → forward potential enstrophy cascade; PV transport
  - Predator → $|\phi_q|^2 \sim \langle V_{E,\theta}^2 \rangle$ {
    - ⇒ growth of *n=0, m=0* Z.F. by turbulent Reynolds work
    - ⇒ Analogous → inverse energy cascade

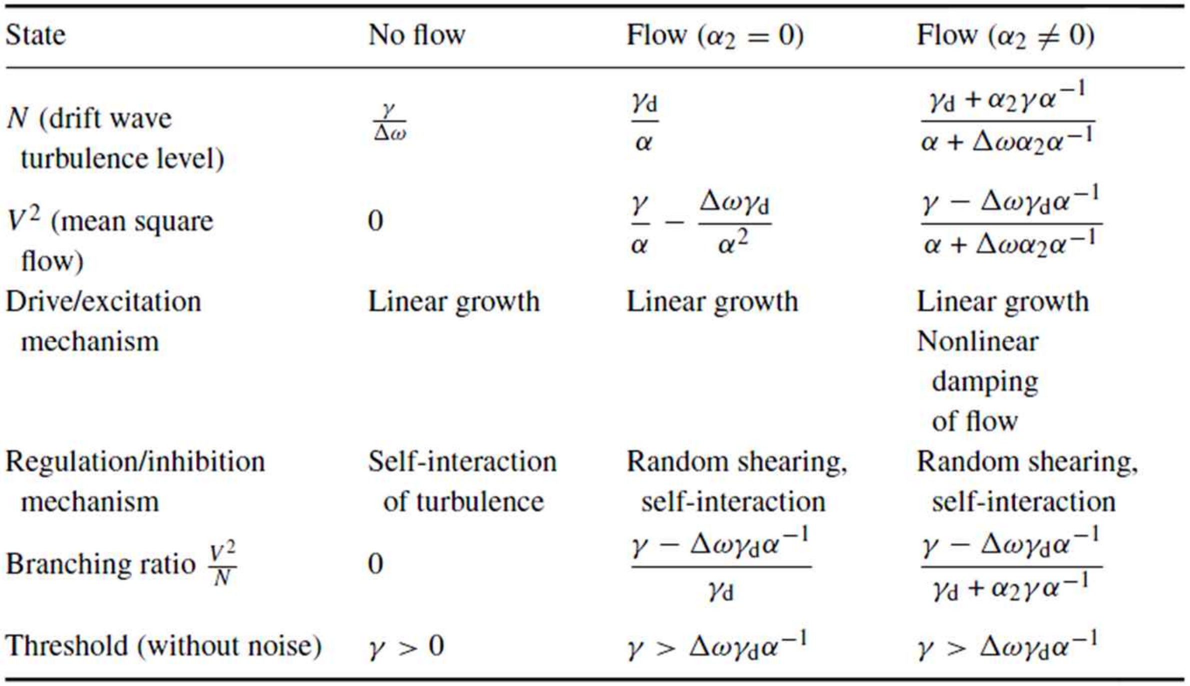
- Mean Field Predator-Prey Model

(P.D. et. al. '94, DI^{2}H '05)

$$\frac{\partial}{\partial t}N = \gamma N - \alpha V^2 N - \Delta\omega N^2$$

$$\frac{\partial}{\partial t}V^2 = \alpha N V^2 - \gamma_d V^2 - \gamma_{NL}(V^2)V^2$$

System Status



State	No flow	Flow ($\alpha_2 = 0$)	Flow ($\alpha_2 \neq 0$)
N (drift wave turbulence level)	$\frac{\gamma}{\Delta\omega}$	$\frac{\gamma_d}{\alpha}$	$\frac{\gamma_d + \alpha_2\gamma\alpha^{-1}}{\alpha + \Delta\omega\alpha_2\alpha^{-1}}$
V^2 (mean square flow)	0	$\frac{\gamma}{\alpha} - \frac{\Delta\omega\gamma_d}{\alpha^2}$	$\frac{\gamma - \Delta\omega\gamma_d\alpha^{-1}}{\alpha + \Delta\omega\alpha_2\alpha^{-1}}$
Drive/excitation mechanism	Linear growth	Linear growth	Linear growth Nonlinear damping of flow
Regulation/inhibition mechanism	Self-interaction of turbulence	Random shearing, self-interaction	Random shearing, self-interaction
Branching ratio $\frac{V^2}{N}$	0	$\frac{\gamma - \Delta\omega\gamma_d\alpha^{-1}}{\gamma_d}$	$\frac{\gamma - \Delta\omega\gamma_d\alpha^{-1}}{\gamma_d + \alpha_2\gamma\alpha^{-1}}$
Threshold (without noise)	$\gamma > 0$	$\gamma > \Delta\omega\gamma_d\alpha^{-1}$	$\gamma > \Delta\omega\gamma_d\alpha^{-1}$

IV) The Central Question: Secondary Pattern Selection ?!

- Two secondary structures suggested
  - Zonal flow → quasi-coherent, regulates transport via shearing
  - Avalanche → stochastic, induces extended transport events
- Both flux driven… by relaxation
- Nature of co-existence??
- Who wins? Does anybody win?

B) Pattern Competition:

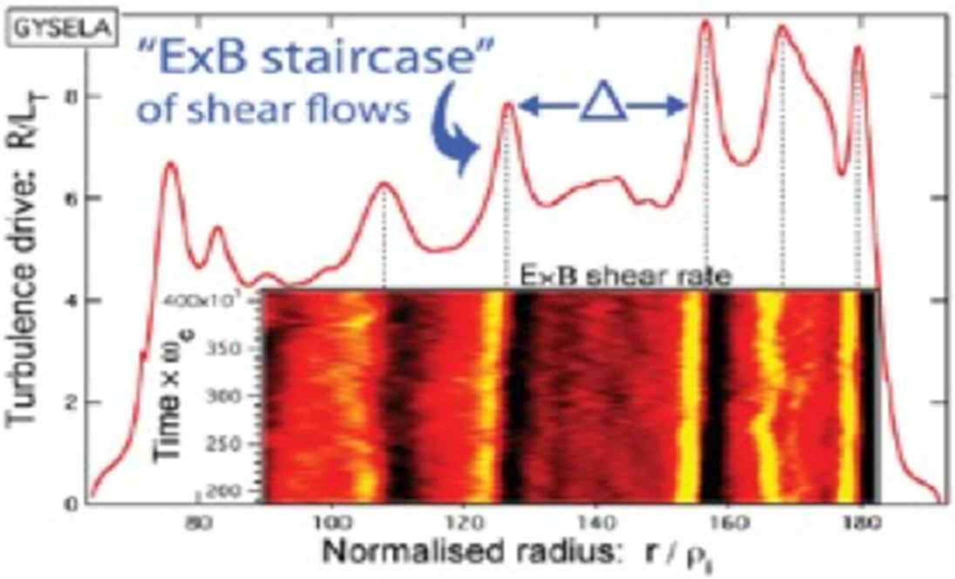
Enter the Staircase….

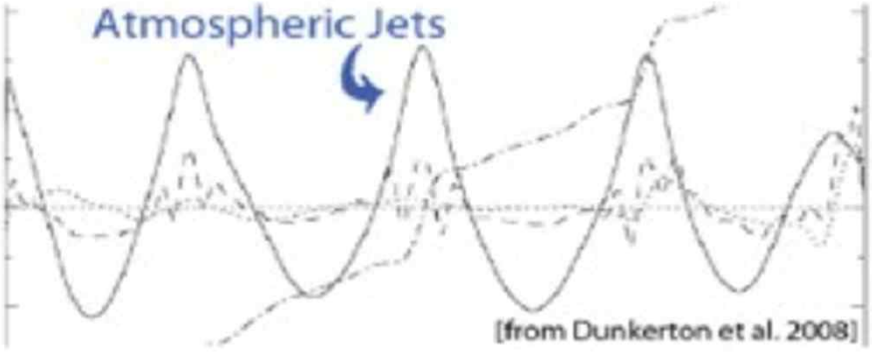
Motivation: ExB staircase formation (1)

- ExB flows often observed to self-organize in magnetized plasmas

  eg. mean sheared flows, zonal flows, ...

- `ExB staircase' is observed to form (G. Dif-Pradalier, P.D. et al. Phys. Rev. E. '10)





- flux driven, full f simulation
- Quasi-regular pattern of shear layers and profile corrugations
- Region of the extent $\Delta \gg \Delta_c$ interspersed by temp. corrugation/ExB jets

  → ExB staircases

- so-named after the analogy to PV staircases and atmospheric jets
- Step spacing → avalanche outer-scale

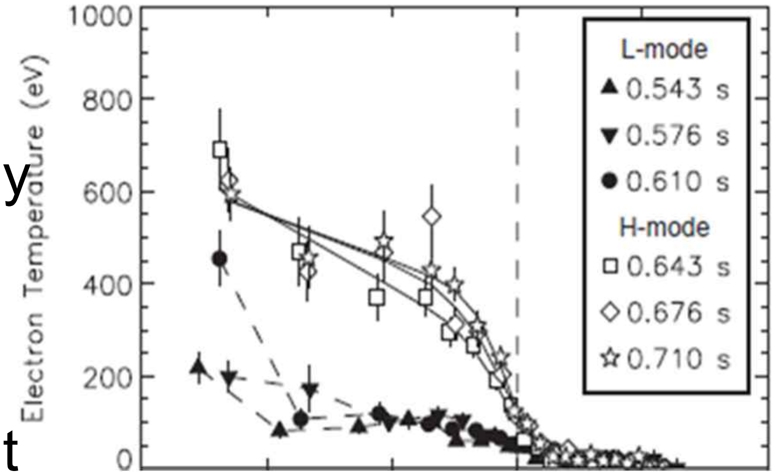
Basic Ideas:
Transport bifurcations and ‘negative diffusion’ phenomena

Transport Barrier Formation (Edge and Internal)

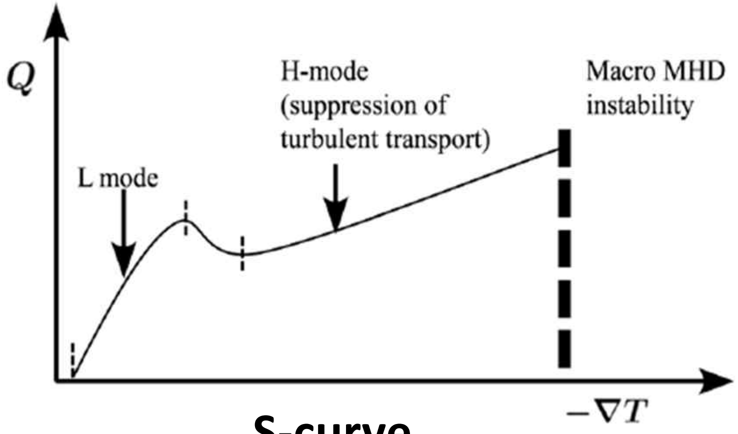
- Observation of ETB formation (L→H transition)
  - THE notable discovery in last 30 yrs of MFE research
  - Numerous extensions: ITB, I-mode, etc.
  - Mechanism: turbulence/transport suppression by ExB shear layers generated by turbulence

- Physics:
  - Spatio-temporal development of bifurcation front in evolving flux landscape
  - Cause of hysteresis, dynamics of back transition

- Fusion:
  - Pedestal width (along with MHD) → ITER ignition, performance
  - ITB control → AT mode
  - Hysteresis + back transition → ITER operation



J.W. Huges et al., PSFC/JA-05-35



S-curve

Why Transport Bifurcation? BDT '90, Hinton '91

- Sheared $V_{E\times B}$ flow quenches turbulence, transport ➔ intensity, phase correlations

- Gradient + electric field ➔ feedback loop (central concept)

i.e. $\vec{E} = \frac{\nabla P_i}{nq} - \vec{V} \times \vec{B}$ ➔ $V_E' = V_E'(\nabla T)$

➔ minimal model $Q = -\frac{\chi(\nabla T)\nabla T}{\left[1+\left(\frac{V_E'}{\omega_{eff}}\right)^2\right]^n} - \chi_{neo}\nabla T$

Residual collisional

turbulent transport
+ shear suppression

n ≡ quenching exponent

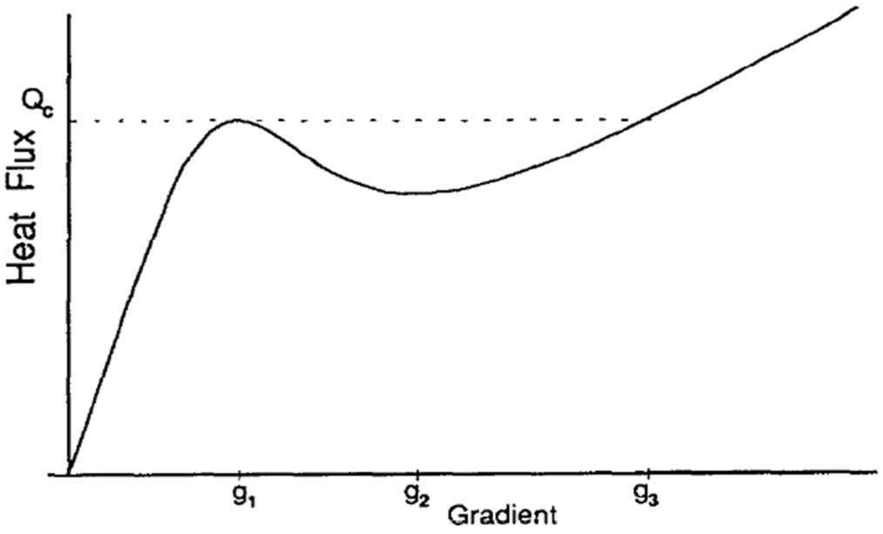
- Feedback:

$$Q \uparrow \rightarrow \nabla T \uparrow \rightarrow V_E' \uparrow \rightarrow (\tilde{n}/n)^2, \chi_T \downarrow$$

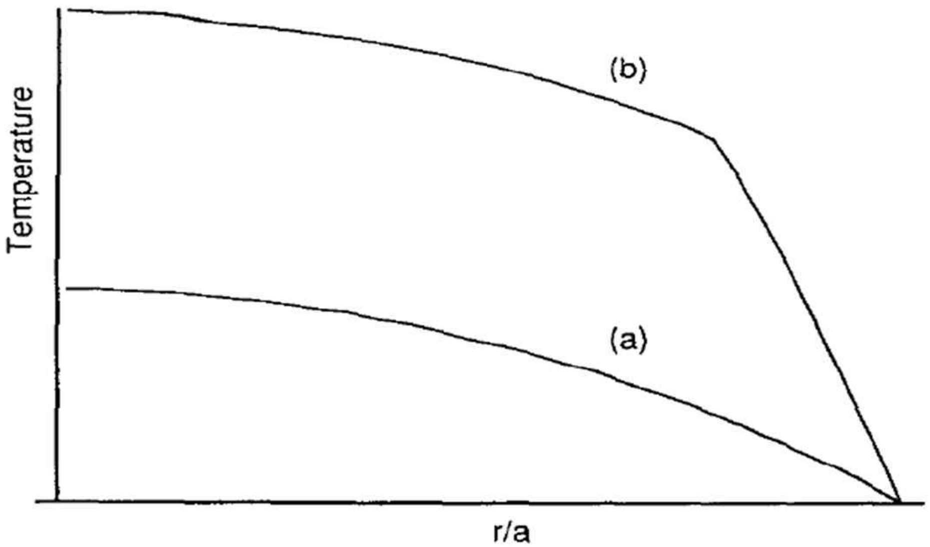
$$\rightarrow \nabla T \uparrow \rightarrow \ldots$$

- Result:

1st order transition (L→H):



Heat flux vs ∇T_i



T profiles

a) L-mode

b) H-mode

- S curve ➔ "negative diffusivity" i.e. $\delta Q \,/\, \delta \nabla T < 0$
- Transport bifurcations observed and intensively studied in MFE since 1982 yet:

➔Little concern with staircases, but if now include modulated ZF feedback on transport?

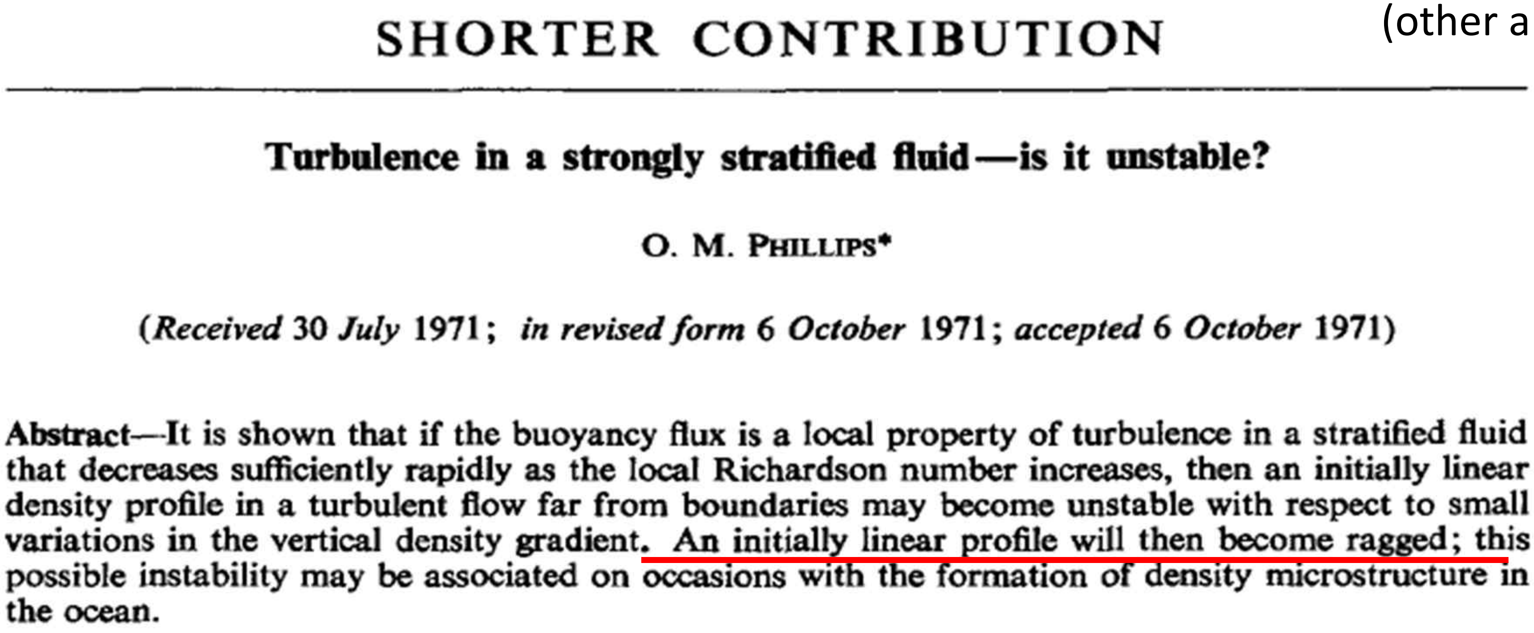
➔ Key questions:

1) Is zonal flow pattern really a staircase? ➔ consequence of inhomogeneous PV mixing induced by modulation?
2) Might observed barriers form via step coalescence in staircases?

Staircase in Fluids

- What is a staircase? – sequence of transport barriers
- Cf Phillips'72:

(other approaches possible)



SHORTER CONTRIBUTION

Turbulence in a strongly stratified fluid—is it unstable?

O. M. PHILLIPS*

(*Received* 30 *July* 1971; *in revised form* 6 *October* 1971; *accepted* 6 *October* 1971)

Abstract—It is shown that if the buoyancy flux is a local property of turbulence in a stratified fluid that decreases sufficiently rapidly as the local Richardson number increases, then an initially linear density profile in a turbulent flow far from boundaries may become unstable with respect to small variations in the vertical density gradient. An initially linear profile will then become ragged; this possible instability may be associated on occasions with the formation of density microstructure in the ocean.

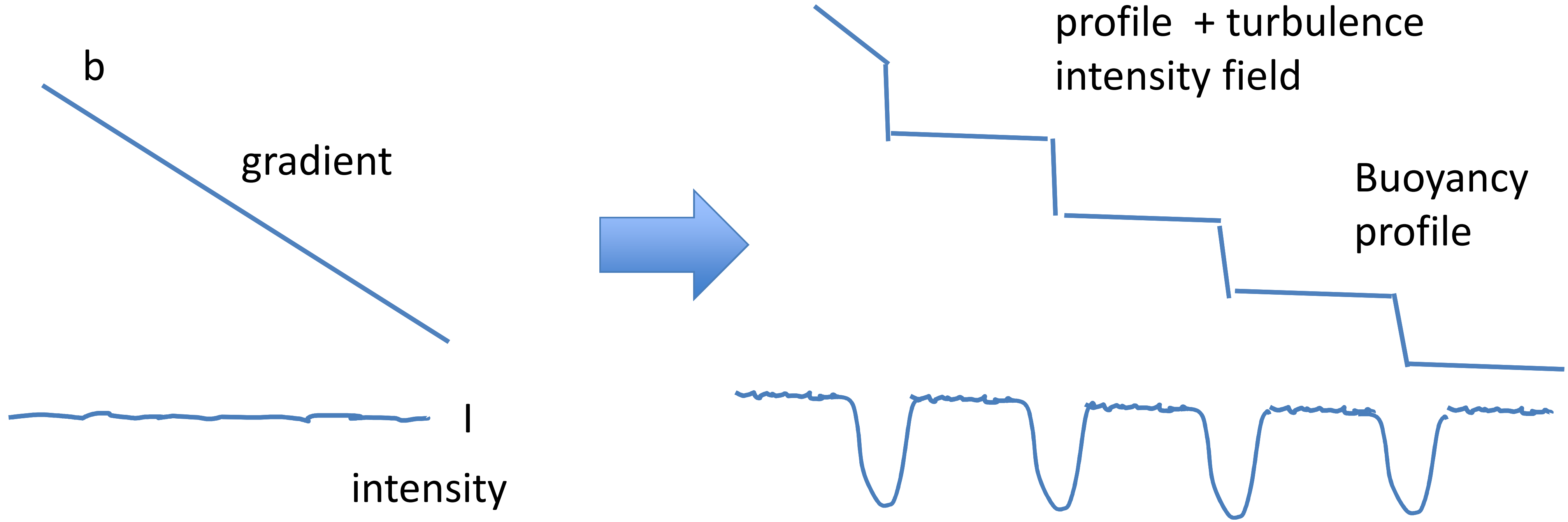
- Instability of mean + turbulence field requiring:

$\delta\Gamma_b/\delta Ri < 0$; flux dropping with increased gradient

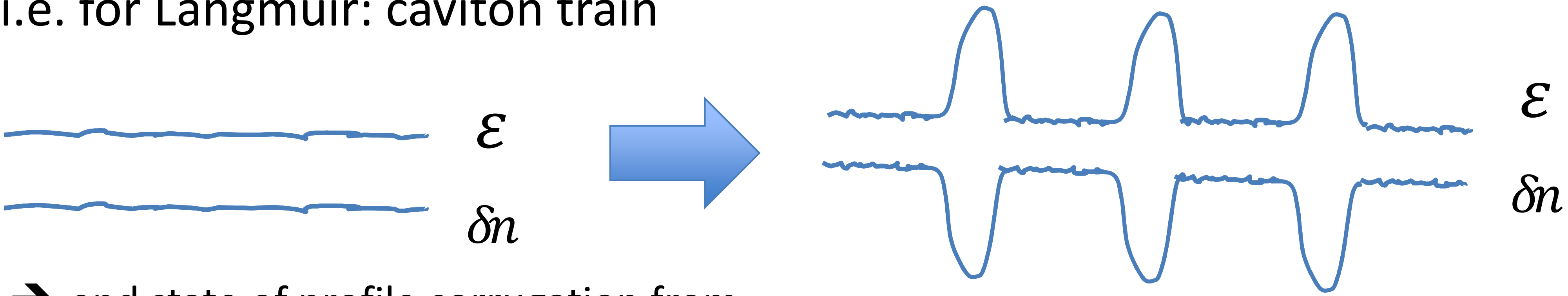
$\Gamma_b = -D_b \nabla b, Ri = g\nabla b/(v')^2$

- Obvious similarity to transport bifurcation

In other words, via modulational instability



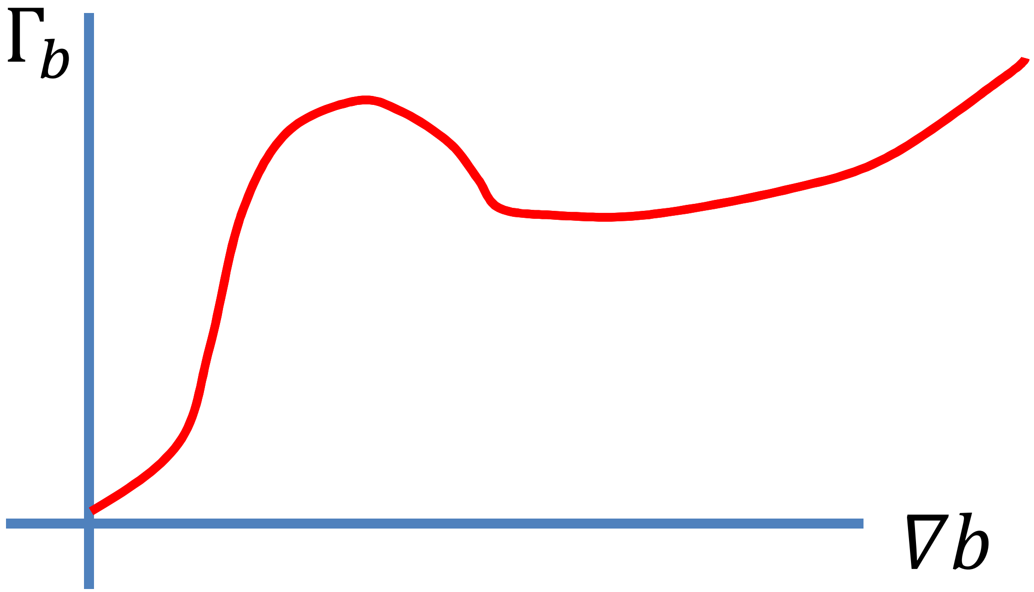
Some resemblance to Langmuir turbulence
i.e. for Langmuir: caviton train



➔ end state of profile corrugation from <u>modulational instability !?</u>

$$\delta n / n \approx -\varepsilon$$

- The physics: “Negative Diffusion” (BLY, ‘98)



“H-mode” like branch
(i.e. residual collisional diffusion)
is not input

- Usually no residual diffusion
- ‘branch’ upswing → nonlinear processes (turbulence spreading)
- If significant molecular diffusion → second branch via collisions

- Instability driven by local transport bifurcation

➔ • $\delta\Gamma_b/\delta\nabla b < 0$

➔ ‘negative diffusion’

Negative slope
Unstable branch

➔ • Feedback loop $\Gamma_b \downarrow \rightarrow \nabla b \uparrow \rightarrow l \downarrow \rightarrow \Gamma_b \downarrow$

Critical element:
$l \rightarrow$ mixing length

- OK: Is there a “simple model” encapsulating the ideas?
- Balmforth, Llewellyn-Smith, Young 1998 → staircase in stirred stably stratified turbulence
- Idea: 1D $K - \epsilon$ model, in lieu W.K.E.
  - turbulence energy; with production, dissipation spreading
  - + Mean field evolution
  - Diffusion: $\tilde{V}\, l_m \sim (\epsilon)^{\frac{1}{2}}\, l_{mix}$
  - l_{mix} → mixing length ?!
  - $\delta\Gamma / \delta\nabla b < 0$ enters via nonlinearity, gradient dependence of length scale

The model

- Mean Field:

$$\partial_t b = \partial_z (D \partial_z b)$$

$$D = e^{1/2} l$$

$$1/l^2 = 1/\, l_f^2 + 1/l_{oz}^2$$

$$e = \langle \tilde{V}^2 \rangle$$

Ozmidov scale

- Fluctuations:

spreading

Production $g\langle \tilde{V} \delta\rho \rangle$

forcing $F \sim \sqrt{e}\,(u_0^2 - e)$

$$\partial_t e = \partial_z D \partial_z e \; - l e^{\frac{1}{2}} \partial_z b \; - \frac{e^{\frac{3}{2}}}{l} + F$$

dissipation

N.B. $\partial_t \left(\int [e - zb\,] \right) = 0$ (energy balance)

- What is l_{mix} ?

$1/\, l^2 = 1/\, l_f^2 + 1/\, l_{oz}^2$

l_{oz} : ~ Ozmidov scale

System mixes at steady state
on scale of energy balance

~ balance of buoyancy production vs. dissipation

i.e. $\tilde{V}^3/l \sim g\langle \tilde{V}\ \delta b \rangle$

$$\delta b \sim \left(\tilde{V}/(\tilde{V}/l)\right)\partial b/\partial z$$

➔ $1/l_{oz} \approx (b_z/e)^{1/2}$

or $V(l)/l \sim N$ ➔ l_{oz}

N.B.: $b_z \uparrow$, $e \downarrow$ $\rightarrow$ $l \downarrow$

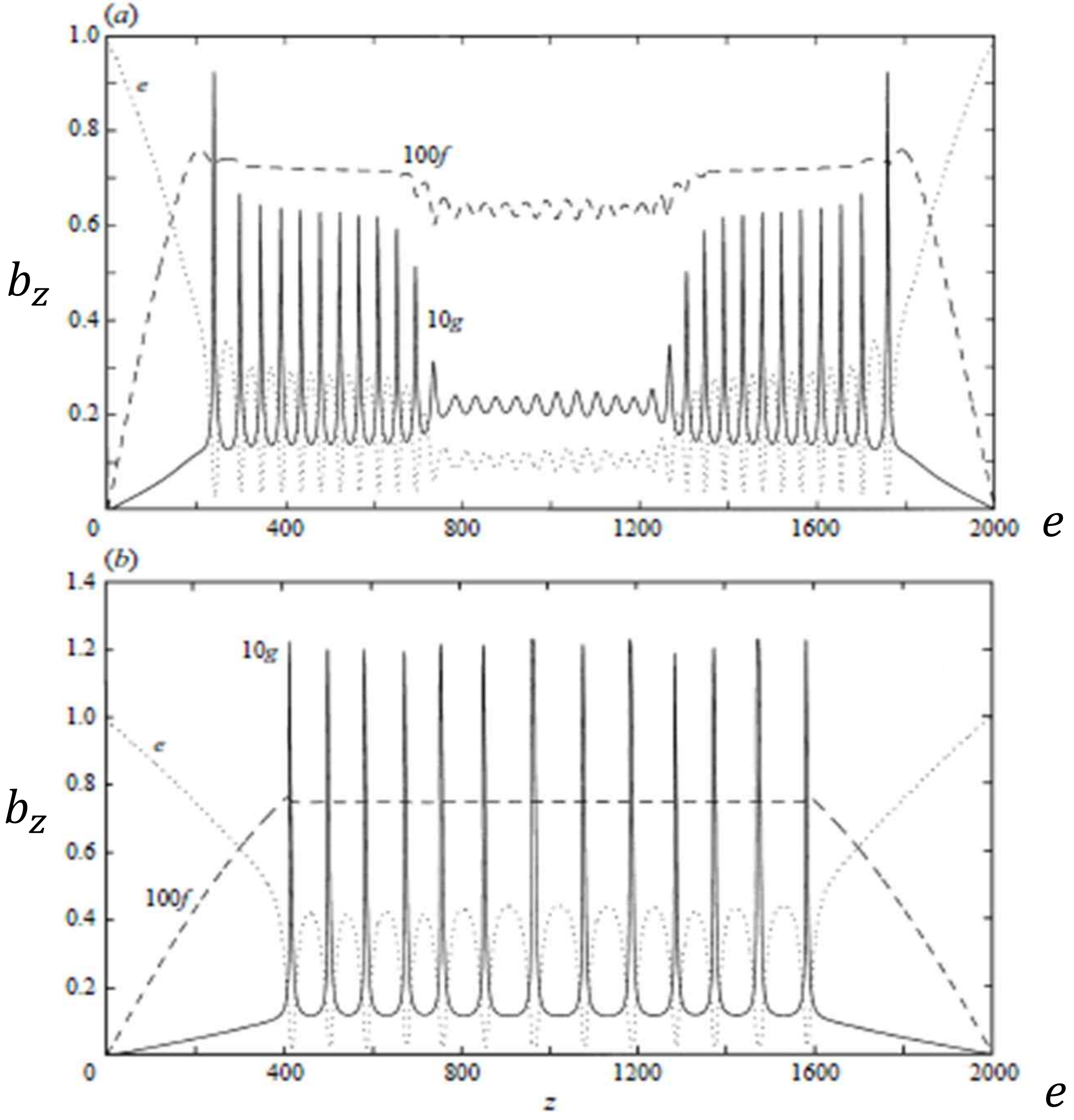
⇓

$e \approx \langle \tilde{V}^2 \rangle$ energy

➔ smallest “stratified” scale

➔ necessary feedback loop

- # A Few Results



Plot of b_z (solid) and e (dotted) at early time. Buoyancy flux is dashed → near constant in core

Later time → more akin expected "staircase pattern". Some <u>condensation</u> into larger scale structures has occurred.

C) Basics: QG Staircase

Staircase in QG Turbulence: A Model

- PV staircases observed in nature, and in the unnatural
- Formulate 'minimal' dynamical model ?! (n.b. Dritschel-McIntyre 2008 does not address dynamics)

Observe:

- 1D adequate: for ZF need 'inhomogeneous PV mixing' + 1 direction of symmetry. Expect ZF staircase
- Best formulate intensity dynamics in terms potential enstrophy $\epsilon = \langle \tilde{q}^2 \rangle$
- Length? : $\Gamma_q \partial \langle q \rangle / \partial y \sim \tilde{q}^3$ (production-dissipation balance)
- → $l \sim \langle \tilde{q}^2 \rangle^{\frac{1}{2}} / \partial \langle q \rangle / \partial y \sim l_{Rhines}$ (i.e. $\omega_{Rossby} \sim k\tilde{v}$)
- Rhines scale is natural length → 'memory' of scale

<u>Model:</u> $\Gamma_q = \langle \tilde{v}_y \tilde{q} \rangle = -D\partial\langle q\rangle/\partial y$ is fundamental quantity (PV flux)

➔ Mean: $\partial_t\langle q\rangle = \partial_y D \partial_y \langle q\rangle$

Dissipation

➔ Potential Enstrophy density: $\partial_t \epsilon - \partial_y D \partial_y \epsilon = D\left(\partial_y\langle q\rangle\right)^2 - \epsilon^{\frac{3}{2}} + F$

Spreading Production Forcing

Where:

$$\frac{1}{l^2} = \frac{1}{l_f^2} + \frac{1}{l_{Rh}^2}$$

$D \sim l^2\sqrt{\epsilon}$ (dimensional)

$$l_{Rh}^2 = \epsilon/\left(\partial_y\langle q\rangle\right)^2$$

$\partial_t\left(\frac{\langle q\rangle^2}{2} + \epsilon\right) = 0$, to forcing, dissipation

$$D_{spr} \approx D_{PV}$$

➔ D → PV mixing

$l_{Rh}(\nabla q)$ ensures inhomogeneity

Alternative Perspective:

- Note: $l^2 = \frac{1}{1+1/l_{Rh}^2} \rightarrow \frac{1}{1+\langle q\rangle'^2/\epsilon}$ $(l_f \sim 1)$
- Reminiscent of weak turbulence perspective:

$$D = D_{pv} = \sum_{\vec{k}} \frac{\langle \tilde{V}^2 \rangle \Delta\omega_{\vec{k}}}{\omega_{\vec{k}}^2 + \Delta\omega_{\vec{k}}^2}$$

$$\omega_{\vec{k}} = -k_x \langle q \rangle' / k^2$$

$$\Delta\omega_{\vec{k}} \approx k\tilde{V}_{\vec{k}}$$

Ala' Dupree'67:

$$D_{pv} \approx \frac{1}{k^2}\left(\sum_{\vec{k}} k^2 \langle \tilde{V}^2 \rangle_{\vec{k}} - \frac{k_x^2 \left(\langle q \rangle'\right)^2}{(k^2)^2}\right)^{1/2}$$

Steeper $\langle q \rangle'$ quenches diffusion → mixing reduced via PV gradient feedback

$$D_{pv} \approx \frac{l_0^2\ \epsilon^{\frac{1}{2}}}{1 + \frac{l_0^2}{\epsilon}(\langle q \rangle')^2} \qquad \leftarrow$$

- ω vs $\Delta\omega$ dependence gives D_{pv} roll-over with steepening
- Rhines scale appears naturally, in feedback strength
- Recovers effectively same model

Physics:

① “Rossby wave elasticity’ (MM) → steeper $\langle q \rangle'$ → stronger memory (i.e. more ‘waves’ vs turbulence)

② Distinct from shear suppression → interesting to dis-entangle

Aside

- What of wave momentum? Austauch ansatz

  Debatable (McIntyre) - but l_{mix} (?)...

- PV mixing ←→ $D\partial_y\langle q\rangle$

  So → $\langle\tilde{V}\tilde{q}\rangle$ → $\partial_y\langle\tilde{V}_y\tilde{V}_x\rangle$ → R.S.

- But:

  R.S. ←→ $\langle k_x k_y\rangle$ ←→ $V_{gy}E$

➔ Feedback:

$$\begin{tikzcd} \langle q\rangle' \uparrow \arrow[r] & l \downarrow \arrow[r] & \epsilon \downarrow \arrow[r] & D \downarrow \arrow[lll, dashed, bend left, "\text{(Production)}"] \end{tikzcd}$$

- Equivalent!
- Formulate in terms mean, Pseudomomentum?

* - Red herring for barriers → l_{mix} quenched

Results:

- Analysis of QG Model Dynamics
- FAQ

- Re-scaled system

$$Q_t = \partial_y \frac{\varepsilon^{1/2}}{(1+Q_y^2\varepsilon)^\kappa}\, Q_y + DQ_{yy}$$ for mean (inhomogeneous PV mixing)

$$\varepsilon_t = \partial_y \frac{\varepsilon^{1/2}}{\left(1+Q_y^2/\varepsilon\right)^\kappa}\, Q_y + L^2 \left\{\frac{Q_y^2}{\left(1+Q_y^2/\varepsilon\right)^\kappa} - \frac{\varepsilon}{\varepsilon_0} + 1\right\} \varepsilon^{1/2} + DQ_{yy}$$

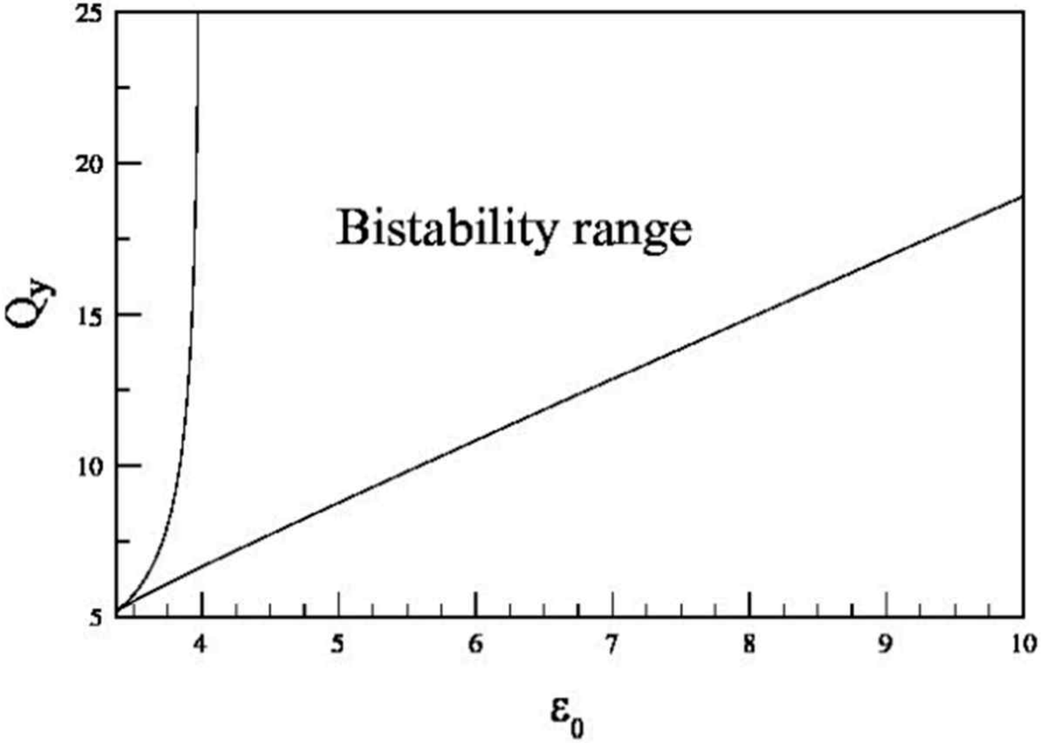
drive

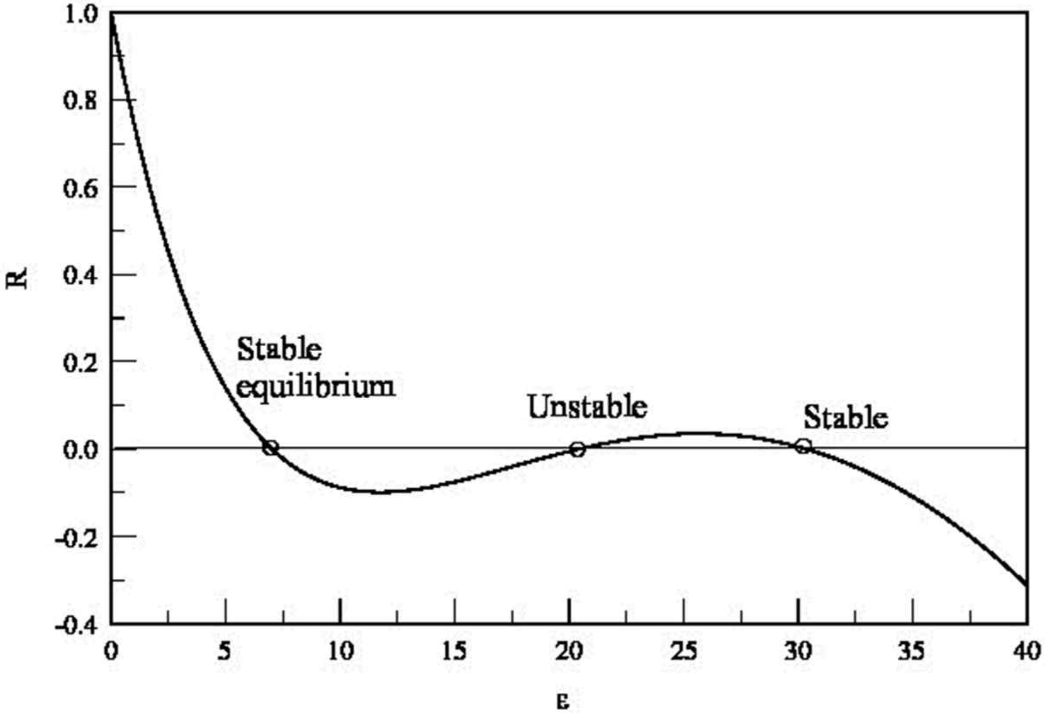
dissipation

(Fluctuation potential enstrophy field)

- Note:
  - Quenching exponent usually $\kappa = 2$ for saturated modulational instability
  - Potential enstrophy conserved to forcing, dissipation, boundary
  - System size L → strength of drive ←→ boundary condition effects!

Structure of RHS: ϵ equation

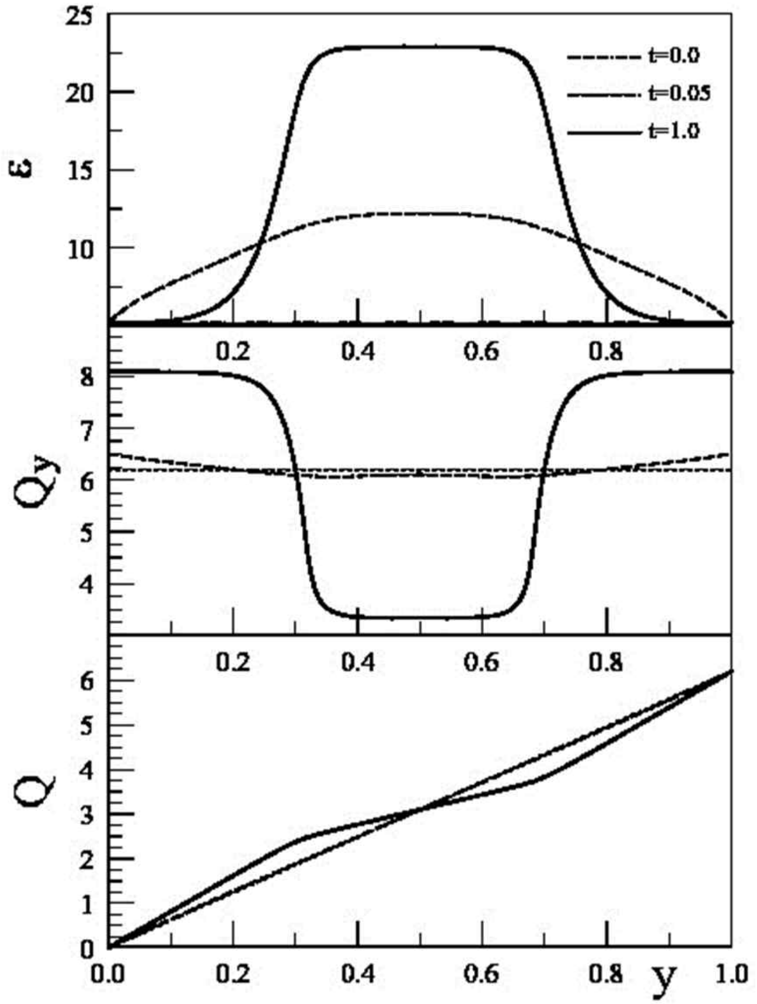


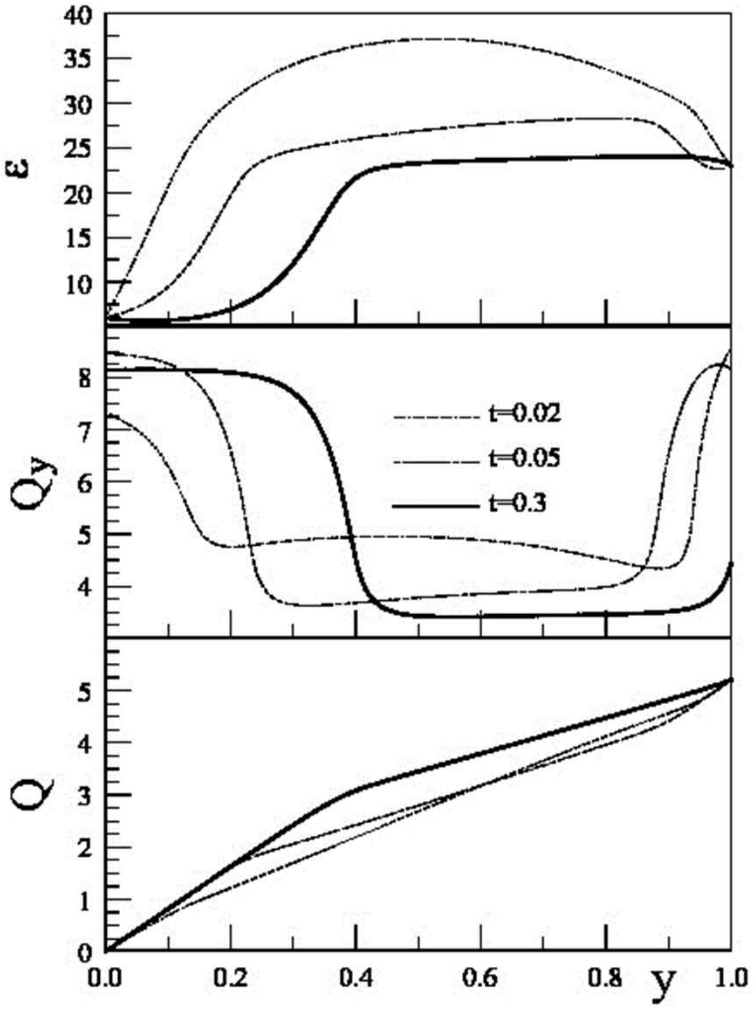


→ Bistability evident

→ Q_y^2 vs ϵ_0 dependencies define range

Basic Results





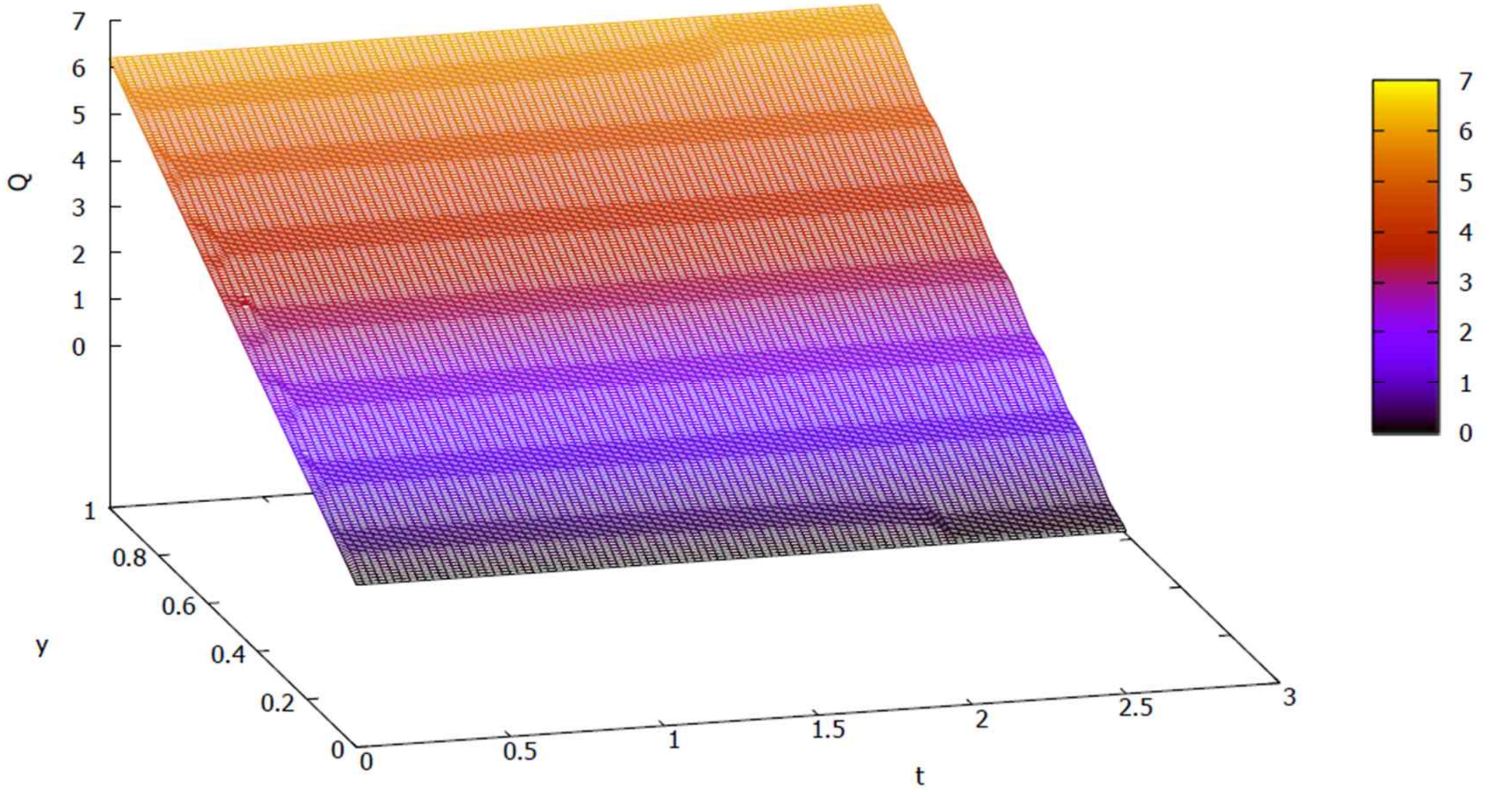
Weak Drive

→ 1 step staircase

→ L^2 increased

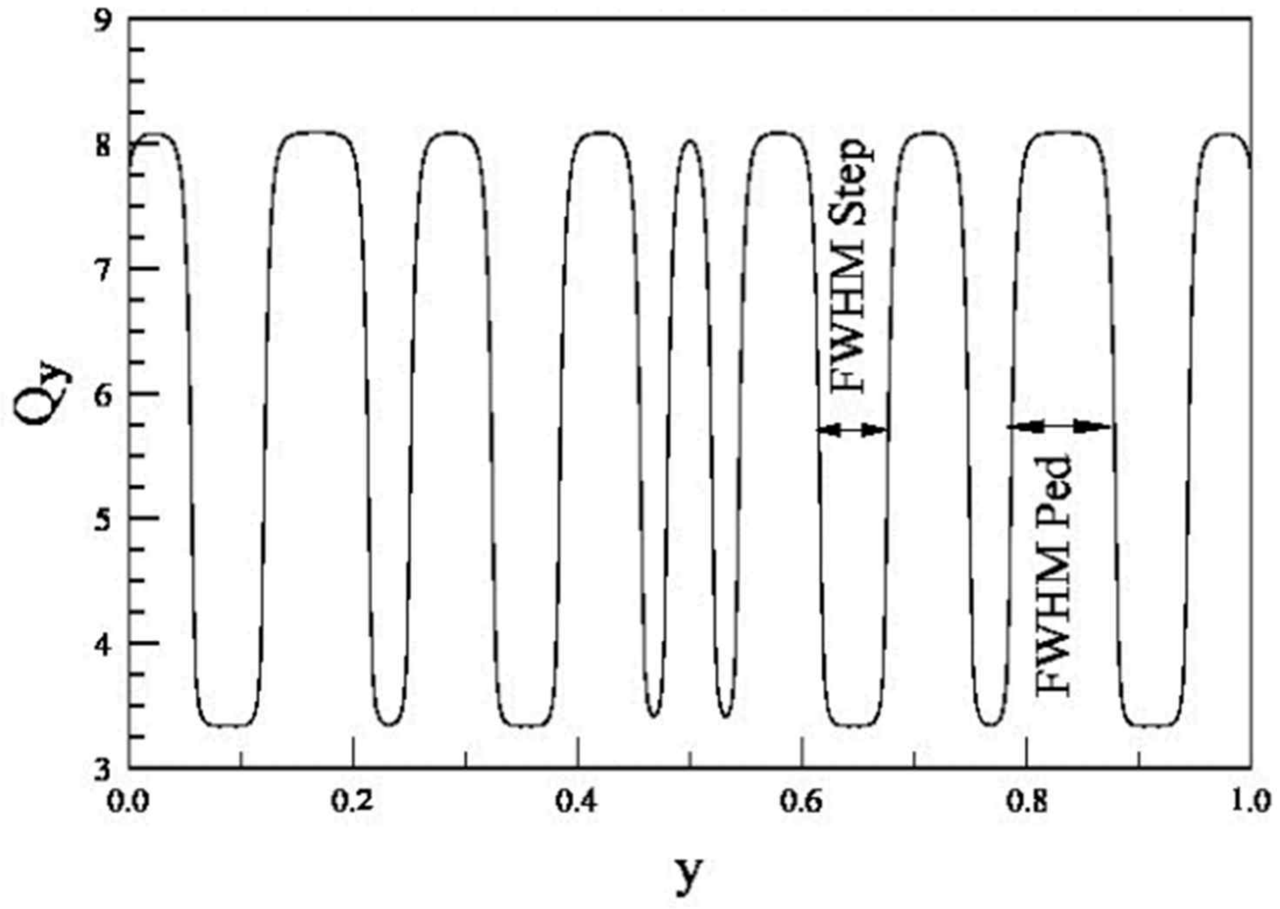
→ Turbulence forces asymmetry

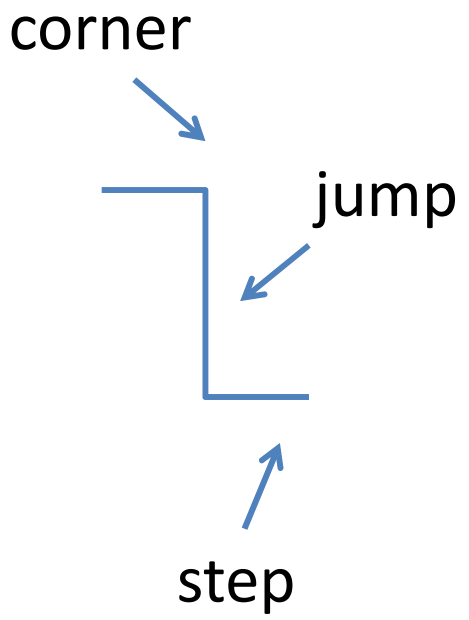
Mergers Occur



- → Surface plot $Q(t, x)$ for Dirichlet
- → 12→7, then persist till 2 layer disappear into wall
- → Further mergers at boundary

Characterization





corner

jump

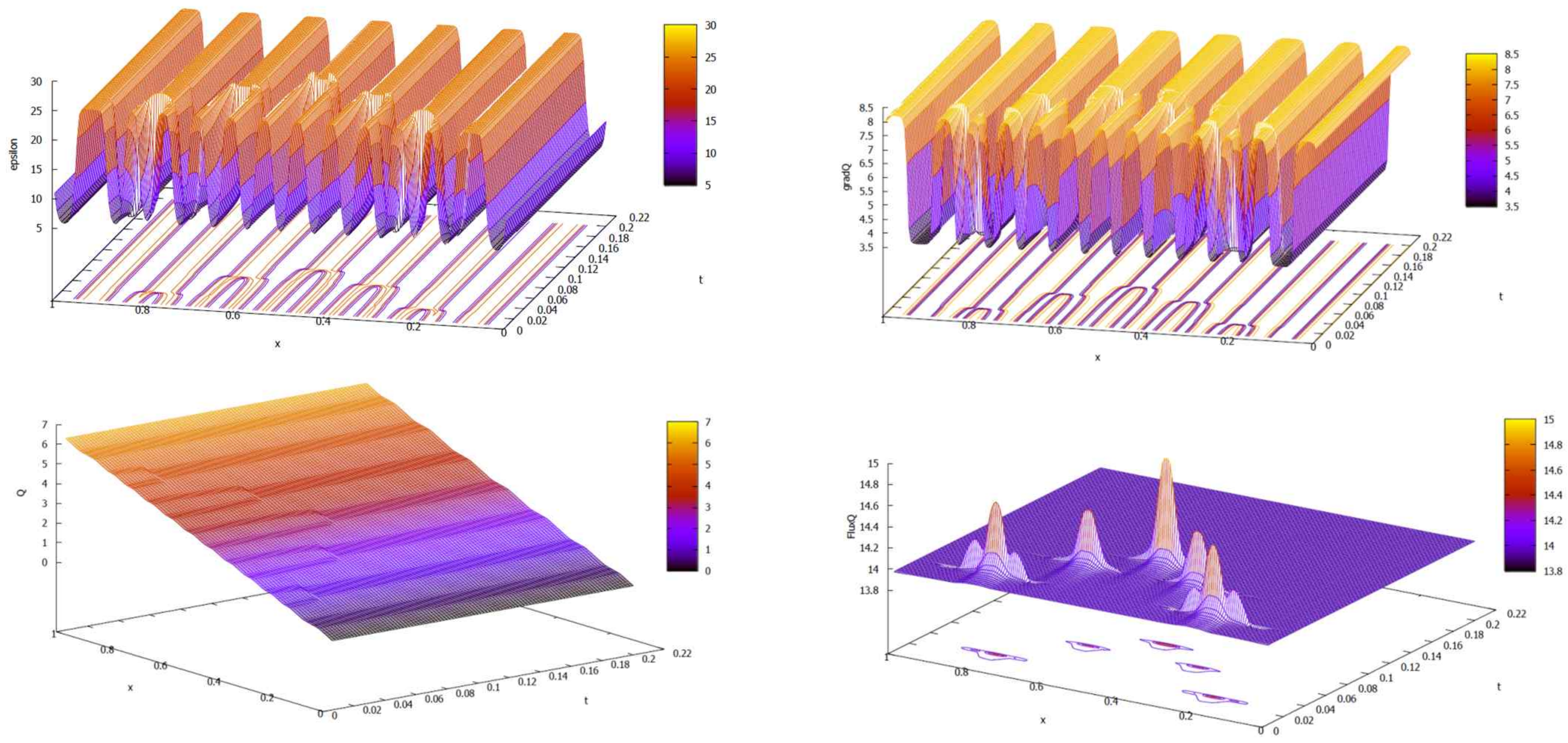
step

- FWHM → jumps/layer

  step

widths

Illustrating the merger sequence



- ϵ
- Q_y

} top

- Q
- Γ_q

} bottom

Note later staircase mergers induce strong flux episodes!

The Hasegawa-Wakatani Staircase:

Profile Structure:

From Mesoscopics → Macroscopics

Extending the Model

Reduced system of evolution Eqs. is obtained from HW system for DW turbulence.

Variables:

$u = \partial_x V_y$ Zonal shearing field

Reduced density: $\log(N/N_0) = \boxed{n(x,t)} + \tilde{n}(x,y,t),$ **Vorticity:** $\rho_s^2 \nabla_\perp^2 \left(e\varphi / T_e\right) = \boxed{u(x,t)} + \tilde{u}(x,y,t)$

Potential Vorticity (PV): $q = n - u,$ **Turbulent Potential Enstrophy (PE):** $\varepsilon = \frac{1}{2}\left\langle (\tilde{n} - \tilde{u})^2 \right\rangle$

Two components

Mean field equations:

density $\partial_t n = -\partial_x \Gamma_n + \partial_x [D_c \partial_x n],$ $\Gamma_n = \left\langle \tilde{v}_x \tilde{n} \right\rangle = -D_n \partial_x n$ ⟶ **Reflect instability**

$\text{Taylor ID}: \ \Pi_u = \left\langle \tilde{v}_x \tilde{u} \right\rangle = \partial_x \left\langle \tilde{v}_x \tilde{v}_y \right\rangle$

vorticity $\partial_t u = -\partial_x \Pi_u + \partial_x [\mu_c \partial_x u],$ $\Pi_u = \left\langle \tilde{v}_x \tilde{u} \right\rangle = (\chi - D_n)\partial_x n - \boxed{\chi}\partial_x u$

Residual vort. flux **Turb. viscosity**

Turbulence evolution:

From closure

$$\partial_t \varepsilon = \partial_x [D_\varepsilon \partial_x \varepsilon] - (\Gamma_n - \Gamma_u)[\partial_x (n - u)] - \varepsilon_c^{-1} \varepsilon^{3/2} + P$$

Turbulence spreading

Internal production

dissipation

External production $\sim \gamma\varepsilon$

Two fluxes Γ_n, Γ_u set model

What is new in this model?

- **In this model PE conservation is a central feature.**
- **Mixing of Potential Vorticity (PV) is the fundamental effect regulating the interaction between turbulence and mean fields.**
- **We use dimensional arguments to obtain functional forms for the turbulent diffusion coefficients. From the QL relation for HW system we obtain**

$$D_n \cong l^2 \frac{\varepsilon}{\alpha} \qquad \chi \cong c_\chi l^2 \frac{\varepsilon}{\sqrt{\alpha^2 + a_u u^2}}$$

l **Dynamic mixing length** ↘ **Rhines scale sets**

α **Parallel diffusion rate**

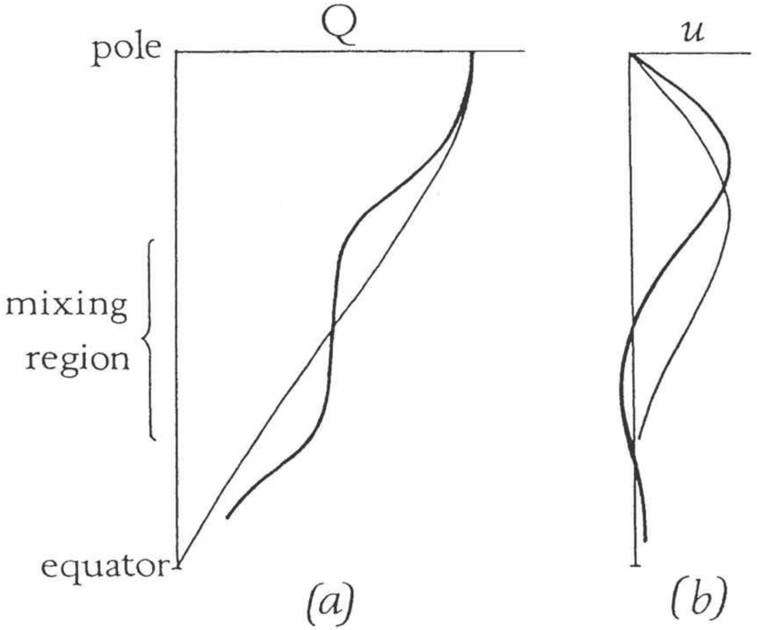
- ***Inhomogeneous mixing of PV results in the sharpening of density and vorticity gradients in some regions and weakening them in other regions, leading to shear lattice and density staircase formation.***

Jet sharpening in stratosphere, resulting from inhomogeneous mixing of PV. (McIntyre 1986)

PV $Q = \nabla^2 \psi + \beta y$

Relative vorticity

Planetary vorticity



Staircase structures

Snapshots of evolving profiles at t=1 (non-dimensional time)

Initial conditions: $n = g_0(1-x), \quad u = 0, \quad \varepsilon = \varepsilon_0$

Boundary conditions: $n(0,t) = g_0,\ n(1,t) = 0; \quad u(0,1;t) = 0; \quad \partial_x \varepsilon(0,1;t) = 0$

Structures:

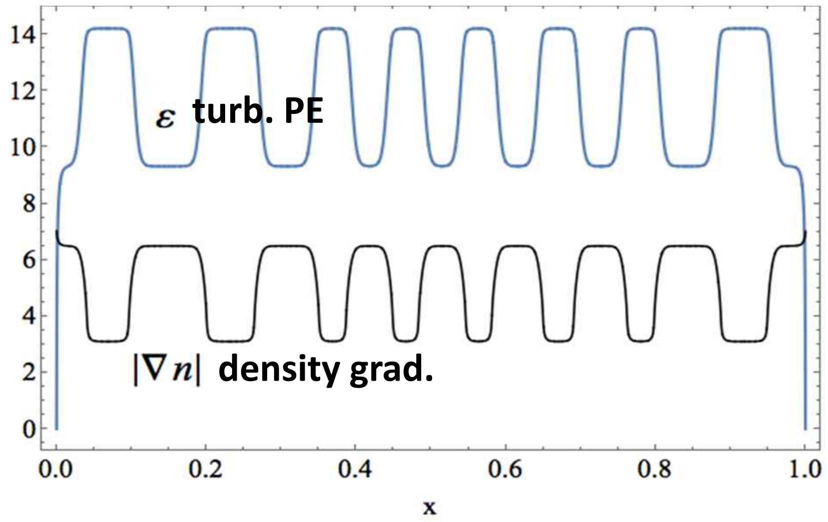
○**Staircase in density profile:**

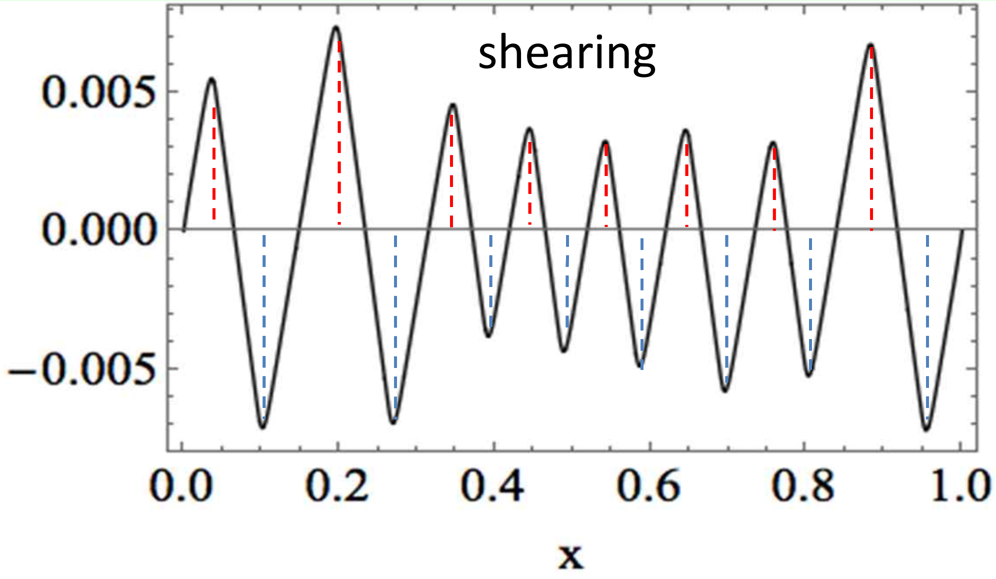
jumps → regions of steepening

steps → regions of flattening

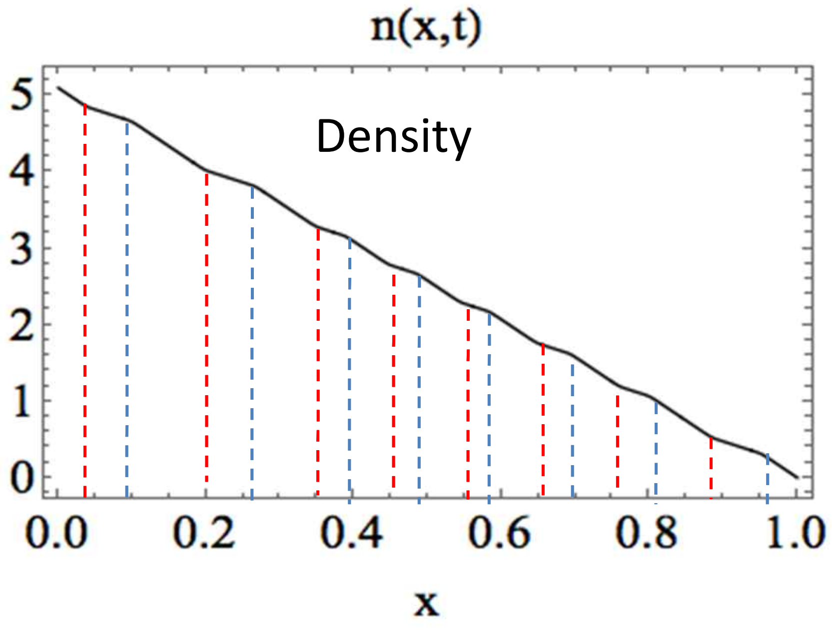
○**At the jump locations, turbulent PE is suppressed.**

○**At the jump locations, vorticity gradient is positive**





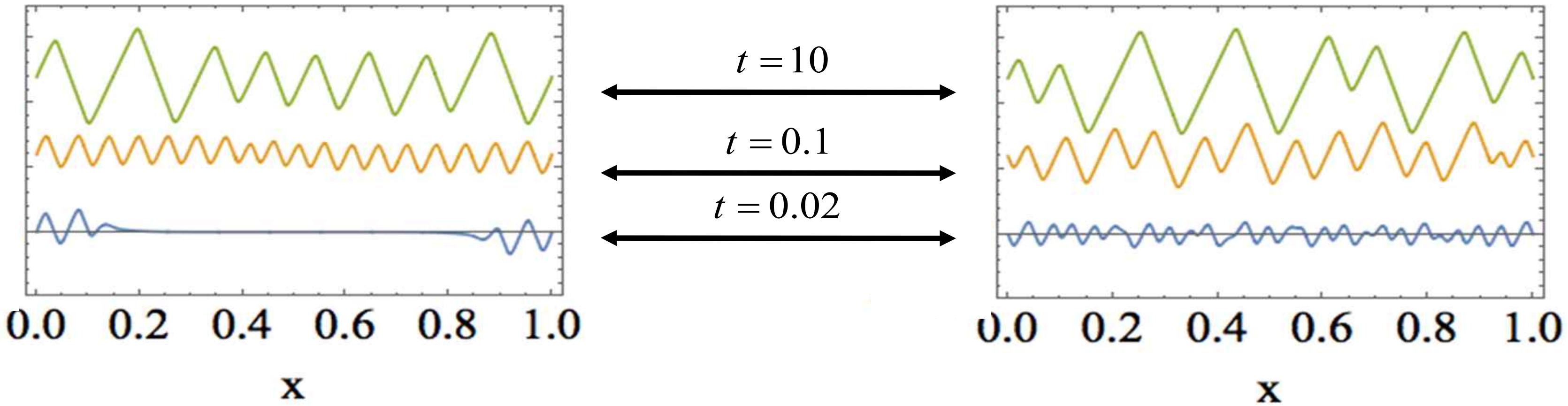
Density
+
Vorticity
lattices



Mergers Occur

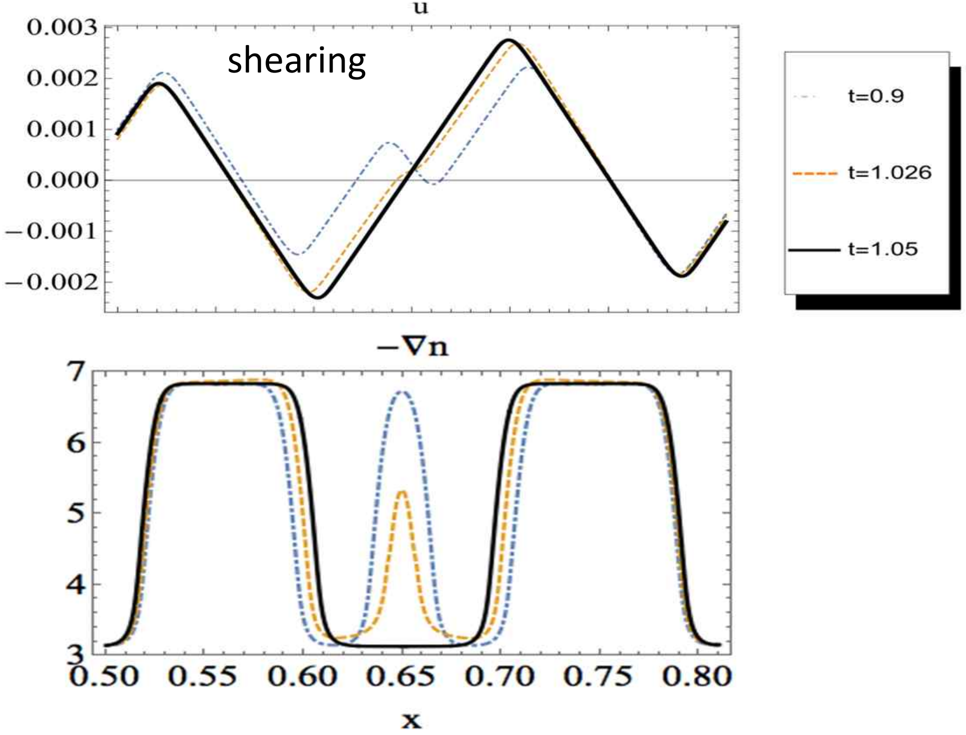
Nonlinear features develop from linear instabilities

$\varepsilon(x=0,1)=0$ **u(x,t)** $\partial_x \varepsilon(x=0,1)=0$ **u(x,t)**

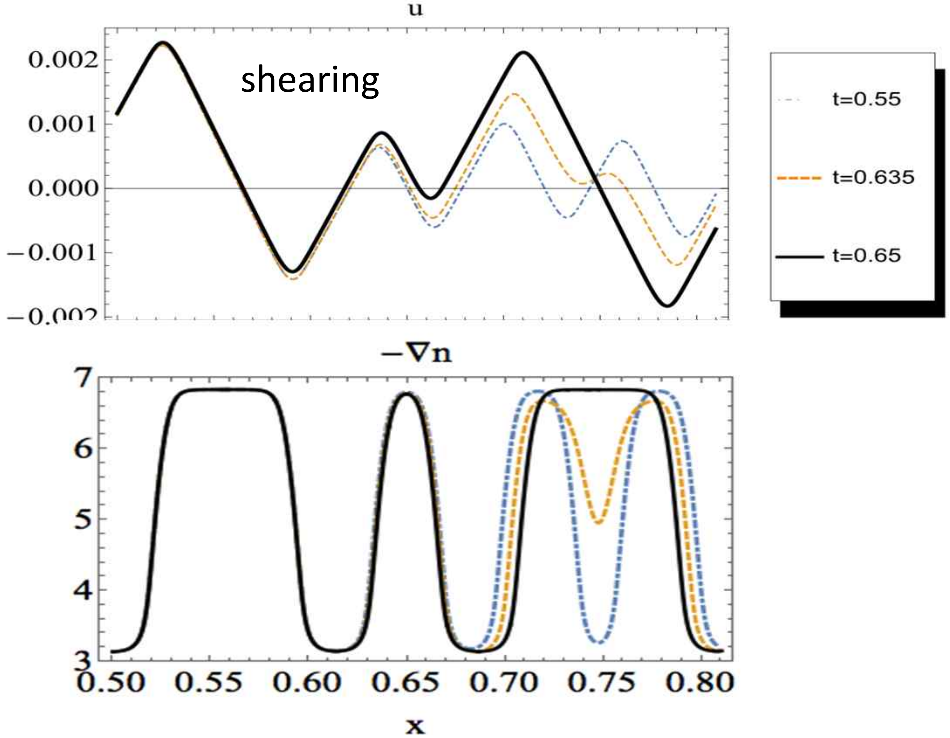


Local profile reorganization: Steps and jumps merge (continues up to times t~O(10))

Merger between steps



Merger between jumps



Shear layer propagation

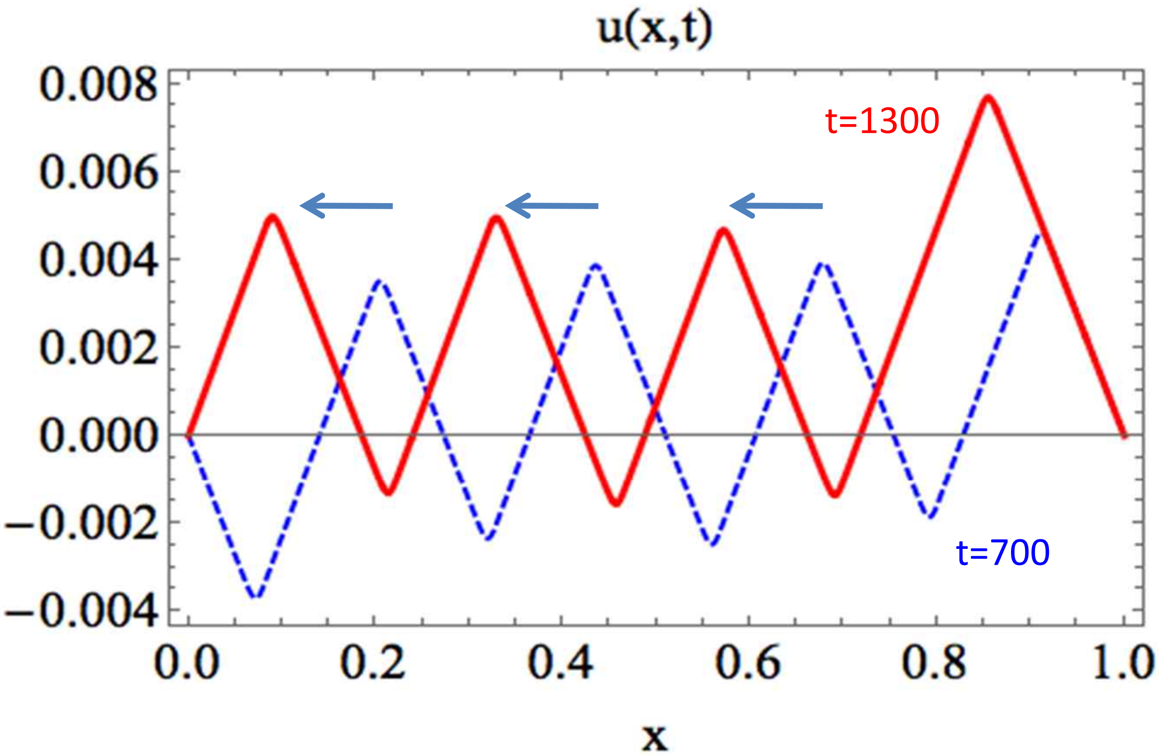
○Shear pattern detaches and delocalizes from its initial position of formation.

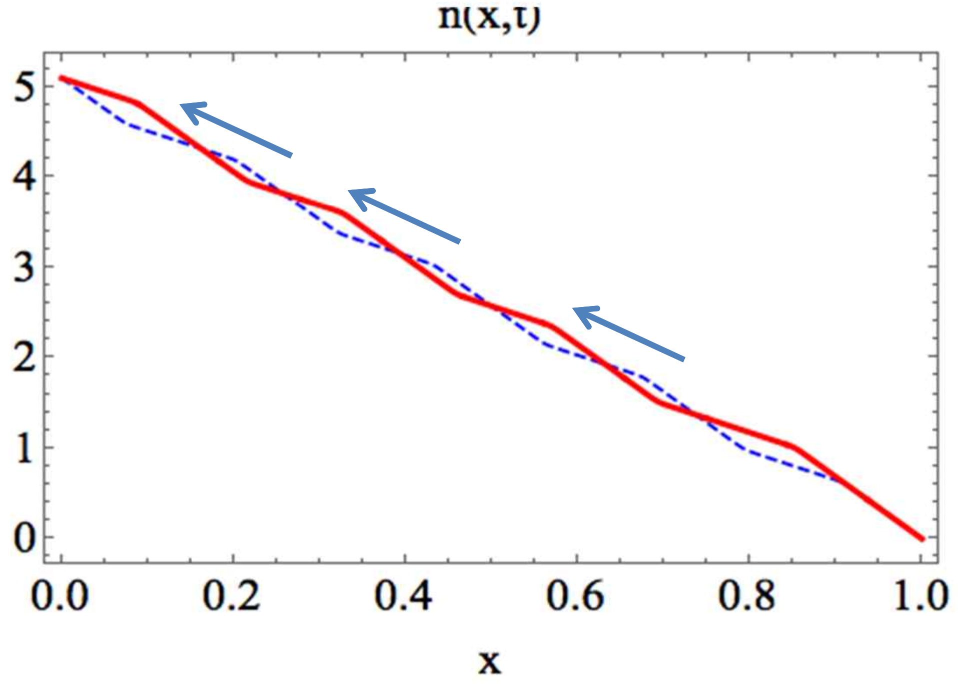
○Mesoscale shear lattice moves in the up-gradient direction. Shear layers condense and disappear at x=0.

○Shear lattice propagation takes place over much longer times. From t~O(10) to t~(10^4).

○Barriers in density profile move upward in an “Escalator-like” motion.

➔ Macroscopic Profile Re-structuring





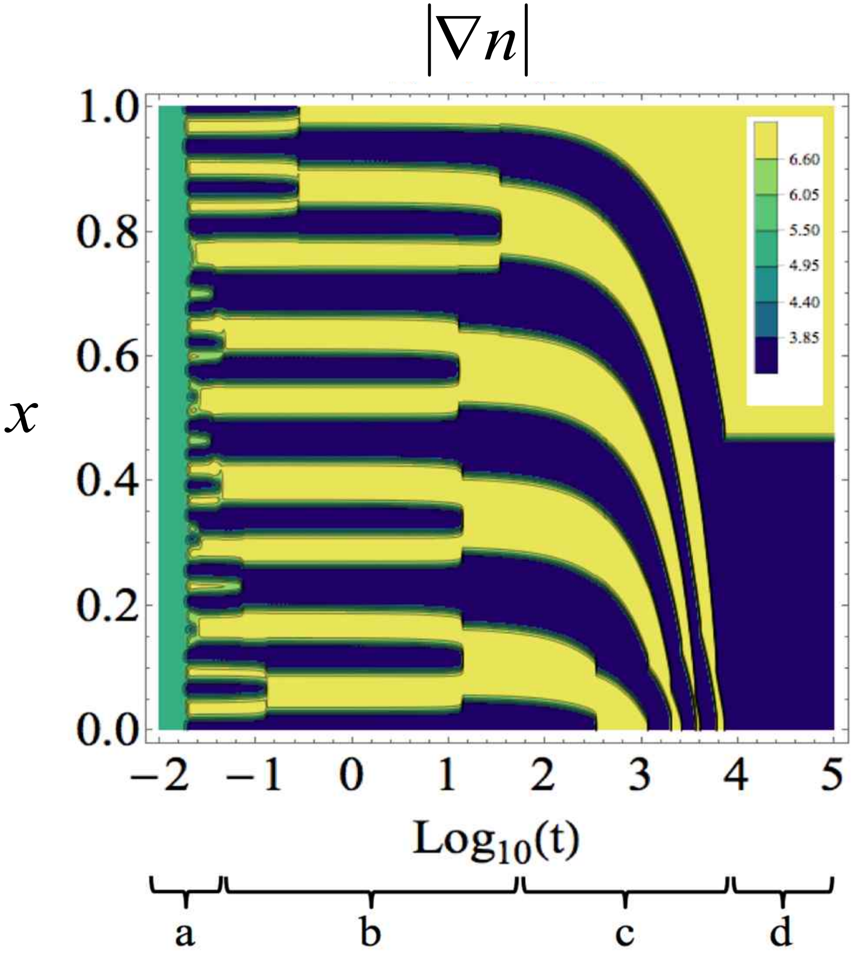
Time evolution of profiles

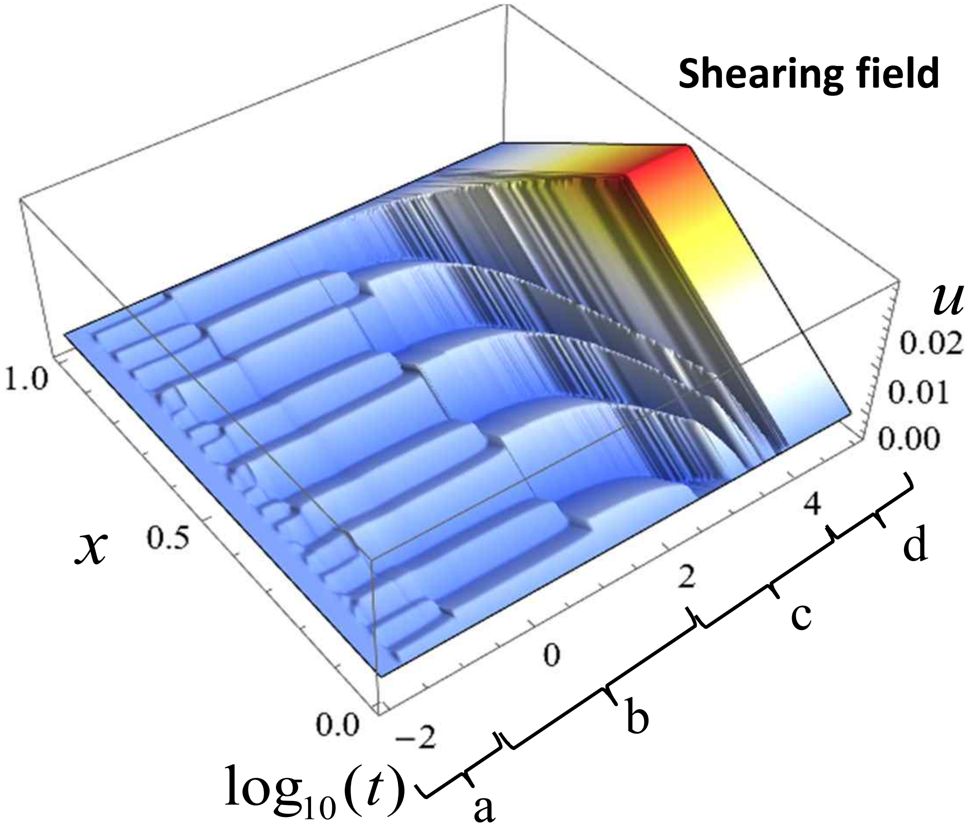
(a) Fast merger of micro-scale SC. Formation of meso-SC.

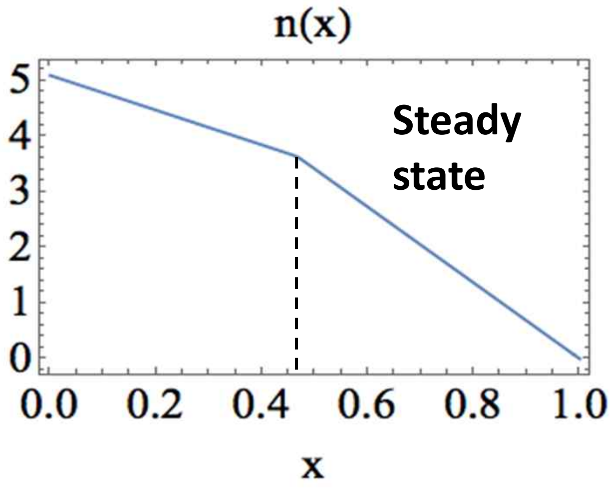
(b) Meso-SC coalesce to barriers

(c) Barriers propagate along gradient, condense at boundaries

(d) Macro-scale stationary profile







Macroscopics: Flux driven evolution

We add an external particle flux drive to the density Eq., use its amplitude Γ_0 as a control parameter to study:

✓What is the mean profile structure emerging from this dynamics?

✓Variation of the macroscopic steady state profiles with Γ_0 (shearing, density, turbulence, and flux).

✓Transport bifurcation of the steady state (macroscopic)

✓Particle flux-density gradient landscape.

$$\partial_t n = -\partial_x \Gamma - \partial_x \Gamma_{dr}(x,t)$$

➔ **Write source as $\nabla \cdot \Pi_{ex}$**

External particle flux (drive) $\Gamma_{dr}(x,t) = \Gamma_0(t)\exp[-x/\Delta_{dr}]$

Internal particle flux (turb. + col.) $\Gamma = -[D_n(\varepsilon,\partial_x q) + D_{col}]\partial_x n$

Transition to Enhanced Confinement can occur

Steady state solution for the system undergoes a transport bifurcation as the flux drive amplitude Γ_0 is raised above a threshold .Γ_{th}

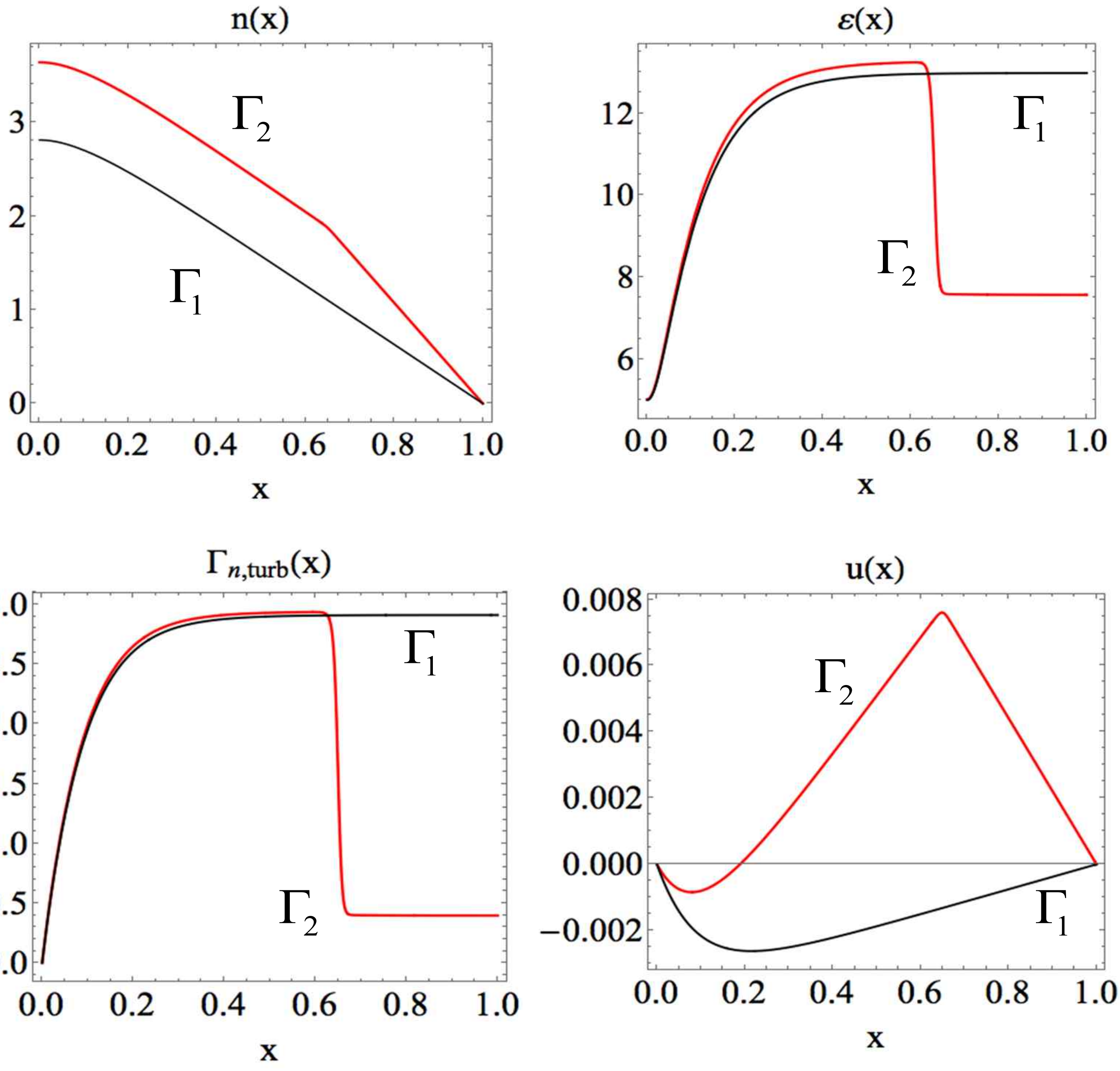
$$\Gamma_1 < \Gamma_{th} < \Gamma_2$$

$$\Gamma_0 = \Gamma_1 \rightarrow \text{Normal Conf. (NC)}$$

$$\Gamma_0 = \Gamma_2 \rightarrow \text{Enhanced Conf. (EC)}$$

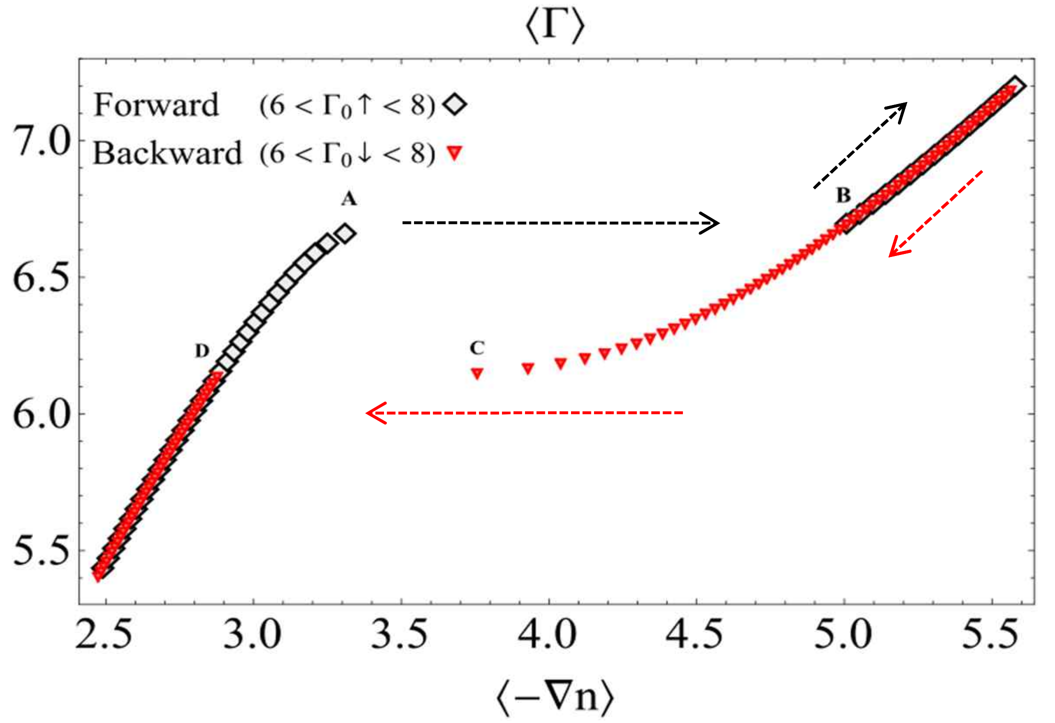
With NC to EC transition we observe:

- Rise in density level
- Drop in turb. PE and turb. particle flux beyond the barrier position
- Enhancement and sign reversal of vorticity (shearing field)



Hysteresis evident in the flux-gradient relation

In one sim. run, from initially flat density profile, Γ_0 is adiabatically raised and lowered back down again.



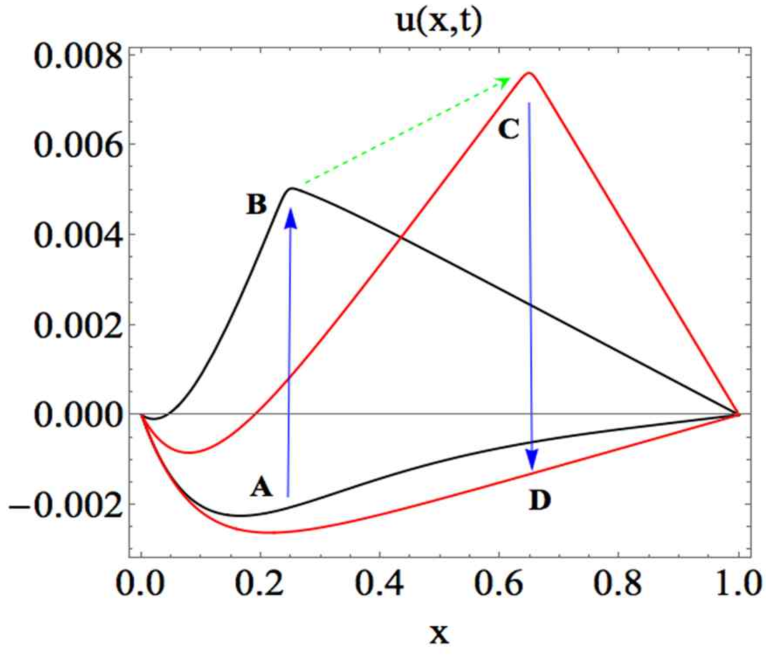
Forward Transition:
Abrupt transition from NC to EC (from A to B). During the transition the system is not in quasi-steady state.

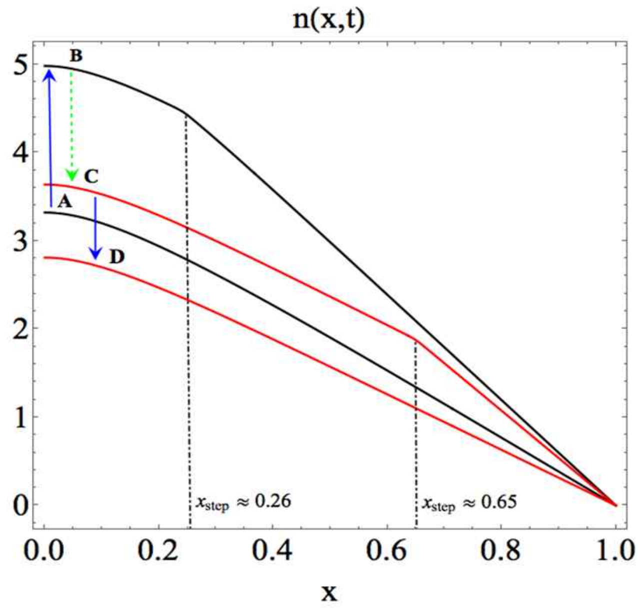
From B to C:
We have continuous control of the barrier position Barrier moves to the right with lowering the density gradient.

Backward Transition:
Abrupt transition from EC to NC (from C to D). Barrier moves rapidly to the right boundary and disappears. system is not in quasi-steady.

$$\langle \Gamma \rangle = \int_0^1 \Gamma(x)dx$$

$$\langle -\partial_x n \rangle = \int_0^1 [-\partial_x n(x,t)]dx$$

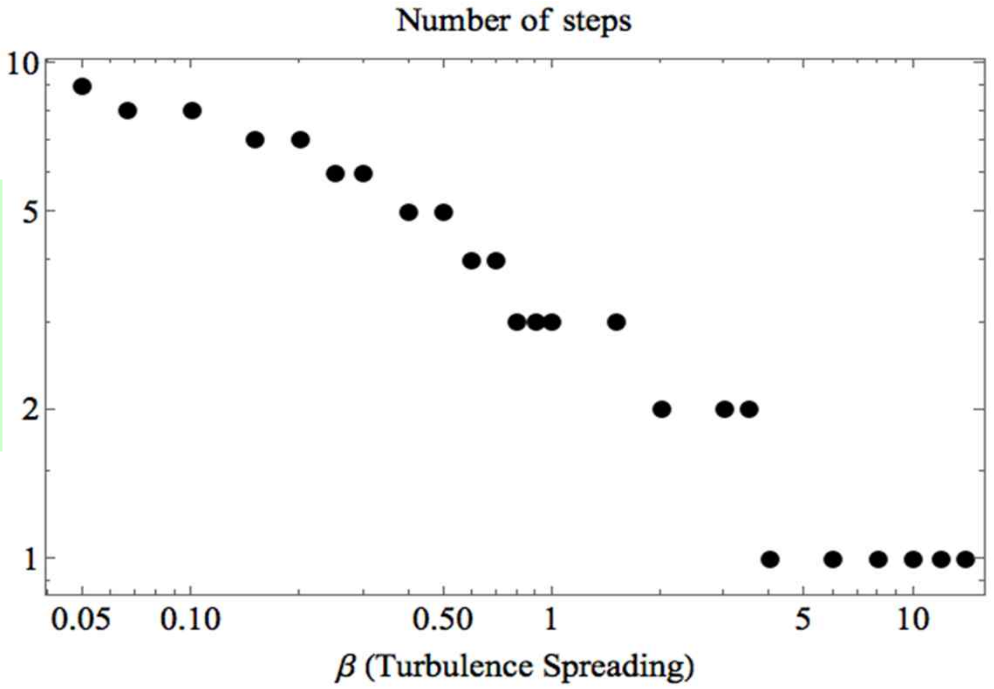




Role of Turbulence Spreading

○ **Large turbulence spreading wipes out features on smaller spatial scales in the mean field profiles, resulting in the formation of smaller number density and vorticity jumps.**

$$\partial_t \varepsilon = \beta \partial_x [(l^2 \varepsilon^{1/2}) \partial_x \varepsilon] + \dots$$



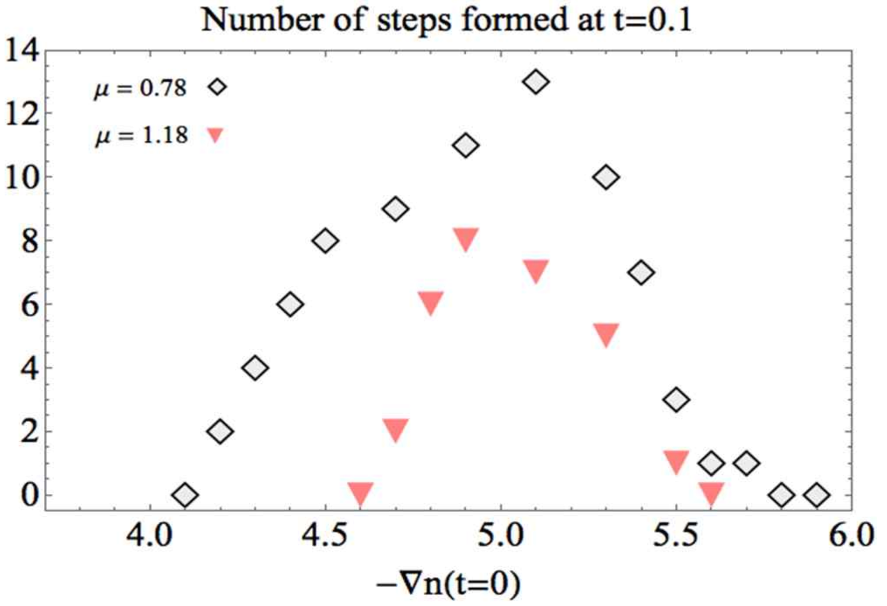
Initial condition dependence

○**Solutions are not sensitive to initial value of turbulent PE.**

○**Initial density gradient is the parameter influencing the subsequent evolution in the system.**

○**At lower viscosity more steps form.**

○**Width of density jumps grows with the initial density gradient.**



E) Conclusions and Lessons

→ Towards a Better Model

Lessons

- A) Staircases happen
  - Staircase is ‘natural upshot’ of modulation in bistable/multi-stable system
  - Bistability is a consequence of mixing scale dependence on gradients, intensity ←→ define feedback process
  - Bistability effectively locks in inhomogeneous PV mixing required for zonal flow formation
  - Mergers result from accommodation between boundary condition, drive(L), initial secondary instability
  - Staircase is natural extension of quasi-linear modulational instabilty/predator-prey model

Lessons

- B) Staircases are Dynamic
  - Mergers occur
  - Jumps/steps migrate. B.C.’s, drive all essential.
  - Condensation of mesoscale staircase jumps into macroscopic transport barriers occurs. ➔ Route to barrier transition by global profile corrugation evolution vs usual picture of local dynamics
  - Global 1st order transition, with macroscopic hysteresis occurs
  - Flux drive + B.C. effectively constrain system states.

Status of Theory

- N.B.: Alternative mechanism via jam formation due flux-gradient time delay → see Kosuga, P.D., Gurcan; 2012, 2013
- a) Elegant, systematic WTT/Envelope methods miss elements of feedback, bistability

  b) $K - \epsilon$ genre models crude, though elucidate much
- Some type of synthesis needed
- Distribution of dynamic, nonlinear scales appear desirable
- Total PV conservation demonstrated utility and leverage.

- Are staircase models:
  - Natural solution to “predator-prey” problem domains via decomposition (akin spiondal)?
  - Natural reduced DOF models of profile evolution?
  - Realization of ‘non-local’ dynamics in transport?

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